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EXAMPLES OF FOURIER SERIES

Leif Mejlbro

Examples of Fourier series

Calculus 4c-1

Examples of Fourier series – Calculus 4c-1

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Contents

Introduction

Here we present a collection of examples of applications of the theory of Fourier series. The reader is also referred to *Calculus 4b* as well as to *Calculus 3c-2*.

It should no longer be necessary rigourously to use the ADIC-model, described in *Calculus 1c* and *Calculus 2c*, because we now assume that the reader can do this himself.

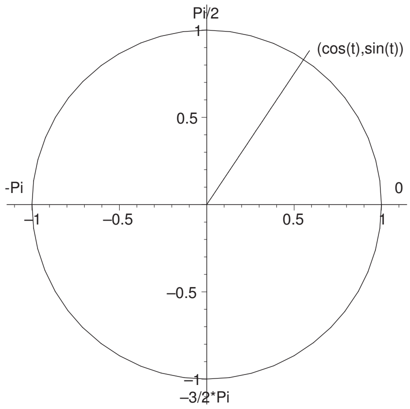
Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
20th May 2008

1 Sum function of Fourier series

A general remark. In some textbooks the formulation of the main theorem also includes the unnecessary assumption that the graph of the function does not have vertical half tangents. It should be replaced by the claim that $f \in L^2$ over the given interval of period. However, since most people only know the old version, I have checked in all examples that the graph of the function does not have half tangents. Just in case ... $\Diamond$

Example 1.1 *Prove that* $\cos n\pi = (-1)^n$, $n \in \mathbb{N}_0$. *Find and prove an analogous expression for* $\cos n\,\dfrac{\pi}{2}$ *and for* $\sin n\,\dfrac{\pi}{2}$.
(Hint: check the expressions for $n = 2p$, $p \in \mathbb{N}_0$, *and for* $n = 2p-1$, $p \in \mathbb{N}$*).*



One may interpret $(\cos t, \sin t)$ as a point on the unit circle.
The unit circle has the length 2π, so by winding an axis round the unit circle we see that $n\pi$ always lies in $(-1,0)$ [rectangular coordinates] for n odd, and in $(1,0)$ for n even.
It follows immediately from the geometric interpretation that

$$\cos n\pi = (-1)^n.$$

We get in the same way that at

$$\cos n\,\frac{\pi}{2} = \begin{cases} 0 & \text{for } n \text{ ulige,} \\ (-1)^{n/2} & \text{for } n \text{ lige,} \end{cases}$$

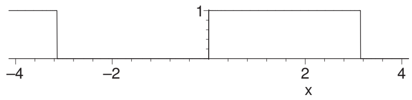
and

$$\sin n\,\frac{\pi}{2} = \begin{cases} (-1)^{(n-1)/2} & \text{for } n \text{ ulige,} \\ 0 & \text{for } n \text{ lige.} \end{cases}$$

Example 1.2 *Find the Fourier series for the function $f \in K_{2\pi}$, which is given in the interval $]-\pi,\pi]$ by*

$$f(t) = \begin{cases} 0 & \text{for } -\pi < t \le 0, \\ 1 & \text{for } 0 < t \le \pi, \end{cases}$$

and find the sum of the series for $t = 0$.



Obviously, $f(t)$ is piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$. Then the adjusted function $f^*(t)$ is defined by

$$f^*(t) = \begin{cases} f(t) & \text{for } t \ne p\pi, \quad p \in \mathbb{Z}, \\ 1/2 & \text{for } t = p\pi, \quad p \in \mathbb{Z}. \end{cases}$$

The Fourier series is pointwise convergent everywhere with the sum function $f^*(t)$. In particular, the sum of the Fourier series at $t = 0$ is

$$f^*(0) = \frac{1}{2}, \qquad \text{(the last question).}$$

The Fourier coefficients are then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} dt = 1,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}\cos nt\,dt = \frac{1}{n\pi}[\sin nt]_0^{\pi} = 0,\ n \geq 1,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi}\sin nt\,dt = -\frac{1}{n\pi}[\cos nt]_0^{\pi} = \frac{1-(-1)^n}{n\pi},$$

hence

$$b_{2n} = 0 \qquad \text{og} \qquad b_{2n+1} = \frac{2}{\pi}\cdot\frac{1}{2n+1}.$$

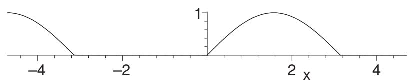
The Fourier series is (with $=$ instead of $\sim$)

$$f^*(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n\cos nt + b_n \sin nt\} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\ \sin(2n+1)t.$$

Example 1.3 *Find the Fourier series for the function* $f \in K_{2\pi}$, *given in the interval* $]-\pi,\pi]$ *by*

$$f(t) = \begin{cases} 0 & \text{for } -\pi < t \leq 0, \\ \sin t & \text{for } 0 < t \leq \pi, \end{cases}$$

and find the sum of the series for $t = p\pi$, $p \in \mathbb{Z}$.



The function f is piecewise C^1 without any vertical half tangents, hence $f \in K^*_{2\pi}$. Since f is continuous, we even have $f^*(t) = f(t)$, so the symbol $\sim$ can be replaced by the equality sign $=$,

$$f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n\cos nt + b_n\sin nt\}.$$

It follows immediately (i.e. the last question) that the sum of the Fourier series at $t = p\pi$, $p \in \mathbb{Z}$, is given by $f(p\pi) = 0$, (cf. the graph).

The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi}\sin t\,dt = \frac{1}{\pi}[-\cos t]_0^{\pi} = \frac{2}{\pi},$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi}\sin t\cdot\cos t\,dt = \frac{1}{2\pi}\left[\sin^2 t\right]_0^{\pi} = 0,$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^\pi \sin t\cdot\cos nt\,dt = \frac{1}{2\pi}\int_0^\pi \{\sin(n+1)t-\sin(n-1)t\}dt\\
&= \frac{1}{2\pi}\left[\frac{1}{n-1}\cos(n-1)t - \frac{1}{n+1}\cos(n+1)t\right]_0^\pi\\
&= \frac{1}{2\pi}\left\{\frac{1}{n-1}\left((-1)^{n-1}-1\right)-\frac{1}{n+1}\left((-1)^{n+1}-1\right)\right\} = -\frac{1}{\pi}\cdot\frac{1+(-1)^n}{n^2-1} \qquad \text{for } n>1.
\end{aligned}$$

Now,

$$1+(-1)^n = \begin{cases} 2 & \text{for } n \text{ even},\\ 0 & \text{for } n \text{ odd},\end{cases}$$

hence $a_{2n+1}=0$ for $n\ge 1$, and

$$a_{2n} = -\frac{2}{\pi}\cdot\frac{1}{4n^2-1},\quad n\in\mathbb{N},\qquad (\text{replace } n \text{ by } 2n).$$

Analogously,

$$b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 t\,dt = \frac{1}{\pi}\cdot\frac{1}{2}\int_0^\pi\{\cos^2 t+\sin^2 t\}dt = \frac{1}{2},$$

and for $n>1$ we get

$$b_n = \frac{1}{\pi}\int_0^\pi \sin t\cdot\sin nt\,dt = \frac{1}{2\pi}\int_0^\pi\{\cos(n-1)t-\cos(n+1)t\}dt = 0.$$

Summing up we get the Fourier series (with =, cf. above)

$$f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^\infty\{a_n\cos nt + b_n \sin nt\} = \frac{1}{\pi}+\frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{4n^2-1}\cos 2nt.$$

Repetition of the last question. We get for $t=p\pi$, $p\in\mathbb{Z}$,

$$f(p\pi) = 0 = \frac{1}{\pi}-\frac{2}{\pi}\sum_{n=1}^\infty\frac{1}{4n^2-1},$$

hence by a rearrangement

$$\sum_{n=1}^\infty \frac{1}{4n^2-1} = \frac{1}{2}.$$

We can also prove this result by a decomposition and then consider the sectional sequence,

$$\begin{aligned}
s_N &= \sum_{n=1}^N\frac{1}{4n^2-1} = \sum_{n=1}^N \frac{1}{(2n-1)(2n+1)}\\
&= \frac{1}{2}\sum_{n=1}^N\left\{\frac{1}{2n-1}-\frac{1}{2n+1}\right\} = \frac{1}{2}\left\{1-\frac{1}{2N+1}\right\}\to\frac{1}{2}
\end{aligned}$$

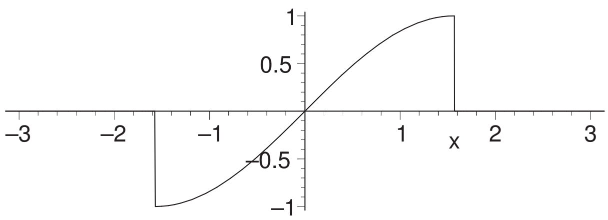
for $N \to \infty$, hence

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{N\to\infty} s_N = \frac{1}{2}.$$

Example 1.4 *Let the periodic function $f : \mathbb{R} \mapsto \mathbb{R}$, of period 2π, be given in the interval $]-\pi, \pi]$ by*

$$f(t) = \begin{cases} 0, & \textit{for } t \in\ ]-\pi, -\pi/2[\,, \\ \sin t, & \textit{for } t \in [-\pi/2, \pi/2]\,, \\ 0 & \textit{for } t \in\ ]\pi/2, \pi]\,. \end{cases}$$

Find the Fourier series of the function and its sum function.



The function f is piecewise C^1 without vertical half tangents, hence $f \in K^*_{2\pi}$. According to the main theorem, the Fourier theorem is then *pointwise convergent* everywhere, and its sum function is

$$f^*(t) = \begin{cases} -1/2 & \text{for } t = -\dfrac{\pi}{2} + 2p\pi, \quad p \in \mathbb{Z}, \\ 1/2 & \text{for } t = \dfrac{\pi}{2} + 2p\pi, \quad p \in \mathbb{Z}, \\ f(t) & \text{ellers.} \end{cases}$$

Since $f(t)$ is discontinuous, the Fourier series *cannot* be uniformly convergent.

Clearly, $f(-t) = -f(t)$, so the function is odd, and thus $a_n = 0$ for every $n \in \mathbb{N}_0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} \sin t \cdot \sin nt\,dt = \frac{1}{\pi}\int_0^{\pi/2} \{\cos((n-1)t) - \cos((n+1)t)\}dt.$$

In the exceptional case $n = 1$ we get instead

$$b_1 = \frac{1}{\pi}\int_0^{\pi/2} (1 - \cos 2t)dt = \frac{1}{\pi}\left[t - \frac{1}{2}\sin 2t\right]_0^{\pi/2} = \frac{1}{2},$$

and for $n \in \mathbb{N} \setminus \{1\}$ we get

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\frac{1}{n-1}\sin((n-1)t) - \frac{1}{n+1}\sin((n+1)t)\right]_0^{\pi/2} \\ &= \frac{1}{\pi}\left\{\frac{1}{n-1}\sin\left(\frac{n-1}{2}\pi\right) - \frac{1}{n+1}\sin\left(\frac{n+1}{2}\pi\right)\right\}. \end{aligned}$$

It follows immediately that if $n > 1$ is odd, $n = 2p+1$, $p \geq 1$, then $b_{2p+1} = 0$ (note that $b_1 = \dfrac{1}{2}$ has been calculated separately) and that (for $n = 2p$ even)

$$\begin{aligned}
b_{2p} &= \frac{1}{\pi}\left\{\frac{1}{2p-1}\,\sin\left(p\pi - \frac{\pi}{2}\right) - \frac{1}{2p+1}\,\sin\left(p\pi + \frac{\pi}{2}\right)\right\} \\
&= \frac{1}{\pi}\left\{\frac{1}{2p-1}\left(-\cos(p\pi)\cdot\sin\frac{\pi}{2}\right) - \frac{1}{2p+1}\left(\cos p\pi\cdot\sin\frac{\pi}{2}\right)\right\} \\
&= \frac{1}{\pi}\,(-1)^{p+1}\left\{\frac{1}{2p-1} + \frac{1}{2p+1}\right\} = \frac{1}{\pi}\,(-1)^{p+1}\cdot\frac{4p}{4p^2-1}.
\end{aligned}$$

By changing variable $p \mapsto n$, it follows that f has the Fourier series

$$f \sim \frac{1}{2}\,\sin t + \sum_{n=1}^{\infty}\frac{1}{\pi}\,(-1)^{n-1}\cdot\frac{4n}{4n^2-1}\,\sin 2nt = f^*(t),$$

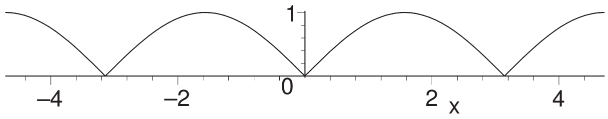
where we already have proved that the series is pointwise convergent with the adjusted function $f^*(t)$ as its sum function.

Example 1.5 *Find the Fourier series for the periodic function $f \in K_{2\pi}$, given in the interval $]\pi, \pi]$ by*

$$f(t) = |\sin t|.$$

Then find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1}.$$



It follows from the figure that f is piecewise differentiable without vertical half tangents, hence $f \in K_{2\pi}^*$. Since f is also continuous, we have $f^*(t) = f(t)$ everywhere. Then it follows by the **main theorem** that the Fourier series is *pointwise convergent* everywhere so we can replace $\sim$ by $=$,

$$f(t) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} \{a_n \cos nt + b_n \sin nt\}.$$

Calculation of the Fourier coefficients. Since $f(-t) = f(t)$ is *even*, we have $b_n = 0$ for every $n \in \mathbb{N}$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin t \cdot \cos nt\, dt = \frac{1}{\pi}\int_0^{\pi} \{\sin(n+1)t - \sin(n-1)t\}dt.$$

Now, $n-1 = 0$ for $n = 1$, so we have to consider this exceptional case separately:

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2t\, dt = \frac{1}{2\pi}[-\cos 2t]_0^{\pi} = 0.$$

We get for $n \neq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \{\sin(n+1)t - \sin(n-1)t\}dt \\ &= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)t + \frac{1}{n-1}\cos(n-1)t\right]_0^{\pi} \\ &= \frac{1}{\pi}\left\{\frac{1+(-1)^n}{n+1} - \frac{1+(-1)^n}{n-1}\right\} = -\frac{2}{\pi}\cdot\frac{1+(-1)^n}{n^2-1}. \end{aligned}$$

Now,

$$1 + (-1)^n = \begin{cases} 2 & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}, \end{cases}$$

so we have to split into the cases of n even and n odd,

$$a_{2n+1} = 0 \quad \text{for } n \geq 1 \quad \text{(and for } n = 0 \text{ by a special calculation)},$$

and

$$a_{2n} = -\frac{4}{\pi} \cdot \frac{1}{4n^2-1} \quad \text{for } n \geq 0, \quad \text{especially } a_0 = +\frac{4}{\pi}.$$

Then the Fourier series can be written with $=$ instead of $\sim$,

$$(1) \quad f(t) = |\sin t| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nt.$$

Remark 1.1 By using a majoring series of the form $c \sum_{n=1}^{\infty} \dfrac{1}{n^2}$, it follows that the Fourier series is uniformly convergent.

We shall find the sum of $\sum_{n=1}^{\infty} (-1)^{n+1}/(4n^2-1)$. When this is compared with the Fourier series, we see that they look alike. We only have to choose t, such that $\cos 2nt$ gives alternatingly ± 1.

By choosing $t = \dfrac{\pi}{2}$, it follows by the pointwise result (1) that

$$f\left(\frac{\pi}{2}\right) = 1 = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos n\pi = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1},$$

thus

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=1-\frac{2}{\pi}=\frac{\pi-2}{\pi},$$

and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}=\frac{\pi-2}{4}.$$

Example 1.6 *Let the periodic function $f:\mathbb{R}\mapsto\mathbb{R}$ of period 2π, be given by*

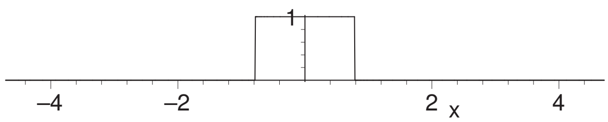
$$f(t)=\begin{cases}0, & \textit{for } t\in\,]-\pi,-\pi/4[\,,\\[6pt] 1, & \textit{for } t\in\,[-\pi/4,\pi/4]\,,\\[6pt] 0 & \textit{for } t\in\,]\pi/4,\pi]\,.\end{cases}$$

1) Prove that f has the Fourier series

$$\frac{1}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{4}\right)\,\cos nt.$$

2) Find the sum of the Fourier series for $t=\dfrac{\pi}{4}$, and then find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{2}\right)$$



Clearly, f is piecewise C^1 (with $f'=0$, where the derivative is defined), hence $f\in K_{2\pi}^*$. According to the main theorem, the Fourier series is then *pointwise convergent everywhere* with the adjusted function as its sum function,

$$f^*(t)=\begin{cases}\dfrac{1}{2} & \text{for } t=\pm\dfrac{\pi}{4}+2p\pi,\quad p\in\mathbb{Z},\\[6pt] f(t) & \text{otherwise.}\end{cases}$$

Since $f(t)$ is *not* continuous, the Fourier series *cannot* be uniformly convergent.

1) Since f is even, we have $b_n = 0$ for every $n \in \mathbb{N}$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi/4} 1\cdot\cos nt\,dt = \frac{2}{\pi n}\,\sin\left(\frac{n\pi}{4}\right)$$

for $n \in \mathbb{N}$. For $n = 0$ we get instead

$$a_0 = \frac{2}{\pi}\int_0^{\pi/4} 1\,dt = \frac{2}{\pi}\cdot\frac{\pi}{4} = \frac{1}{2},$$

so

$$f \sim \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nt = \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{4}\right)\,\cos nt.$$

2) When $t = \dfrac{\pi}{4}$ we get from the beginning of the example,

$$f^*\left(\frac{\pi}{4}\right) = \frac{1}{2} = \frac{1}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{4}\right)\cdot\cos\left(\frac{n\pi}{4}\right) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{2}\right).$$

Then by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{2}\right) = \frac{\pi}{4}.$$

Alternatively,

$$\sum_{n=1}^{\infty}\frac{1}{n}\,\sin\left(\frac{n\pi}{2}\right) = \sum_{p=1}^{\infty}\frac{1}{2p-1}\,\sin\left(p\pi - \frac{\pi}{2}\right) = \sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{2p-1} = \text{ Arctan } 1 = \frac{\pi}{4}.$$

Example 1.7 *Let* $f :]0,2[\mapsto \mathbb{R}$ *be the function given by* $f(t) = t$ *in this interval.*

1) Find a cosine series with the sum $f(t)$ *for every* $t \in]0,2[$.

2) Find a sine series with the sum for every $t \in]0,2[$.

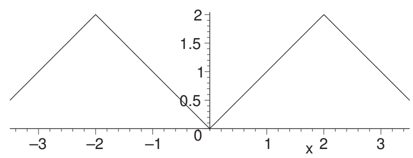
The trick is to extend f as an even, or an odd function, respectively.

1) The *even* extension is

$$F(t) = |t| \quad \text{for } t \in [-2,2], \text{ continued periodically.}$$

It is obviously piecewise C^1 and without vertical half tangents, hence $F \in K_4^*$. The periodic continuation is continuous everywhere, hence it follows by the **main theorem** (NB, a cosine series) with equality that

$$F(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{2}\right),$$



where

$$a_n = \frac{4}{4}\int_0^2 t \cdot \cos\left(n \cdot \frac{2\pi}{4}\, t\right) dt = \int_0^2 t\, \cos\left(\frac{n\pi t}{2}\right) dt, \qquad n \in \mathbb{N}_0.$$

Since we must not divide by 0, we get $n = 0$ as an exceptional case,

$$a_0 = \int_0^2 t\, dt = \left[\frac{t^2}{2}\right]_0^2 = 2.$$

For $n > 0$ we get by partial integration,

$$\begin{aligned} a_n &= \int_0^2 t\, \cos\left(\frac{n\pi}{2}\, t\right) dt = \left[\frac{2}{n\pi}\, t\, \sin\left(\frac{n\pi}{2}\, t\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi}{2}\, t\right) dt \\ &= \frac{4}{\pi^2 n^2}\left[\cos\left(\frac{n\pi}{2}\, t\right)\right]_0^2 = \frac{4}{\pi^2 n^2}\{(-1)^n - 1\}. \end{aligned}$$

For *even* indices $\neq 0$ we get $a_{2n} = 0$.
For *odd* indices we get

$$a_{2n+1} = \frac{2}{\pi^2 (2n+1)^2}\{(-1)^{2n+1} - 1\} = -\frac{8}{\pi^2} \cdot \frac{1}{(2n+1)^2}, \qquad n \in \mathbb{N}_0.$$

The cosine series is then

$$F(t) = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\, \cos\left(n\pi + \frac{\pi}{2}\right) t,$$

and in particular

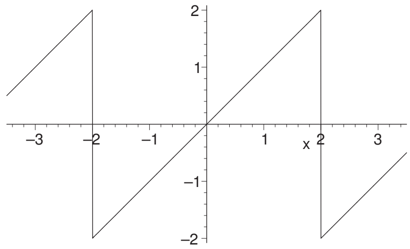
$$f(t) = t = 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\, \cos\left(n + \frac{1}{2}\right)\pi t, \qquad t \in [0, 2].$$

2) The *odd* extension becomes

$$G(t) = t \qquad \text{for } t \in\,]-2, 2[.$$

We adjust by the periodic extension by $G(2p) = 0$, $p \in \mathbb{Z}$. Clearly, $G \in K_4^*$, and since G is odd and adjusted, it follows from the **main theorem** with equality that

$$G(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{2}\right),$$



where

$$b_n = \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt = \frac{2}{n\pi}\left[-t\cos\left(\frac{n\pi t}{2}\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi t}{2}\right) dt$$
$$= \frac{2}{n\pi}\{-2\cos(n\pi)+0\}+0 = (-1)^{n+1}\cdot\frac{4}{n\pi}.$$

The sine series becomes (again with $=$ instead of $\sim$)

$$G(t) = \sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{4}{n\pi}\sin\left(\frac{n\pi t}{2}\right).$$

Thus, in the interval $]0,2[$ we have

$$G(t)=f(t)=t=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,\sin\left(\frac{n\pi t}{2}\right),\qquad t\in\,]0,2[.$$

It is no contradiction that $f(t)=t$, $t\in\,]0,2[$, can be given two different expressions of the same sum.

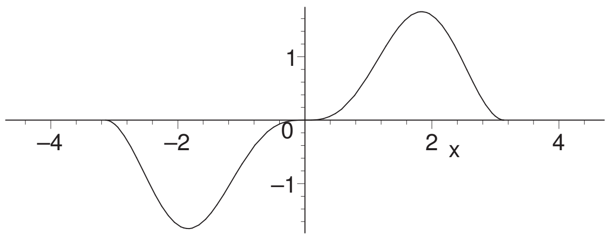
Note that the cosine series is uniformly convergent, while the sine series is *not* uniformly convergent.
3mm
In the applications in the engineering sciences the sine series are usually the most natural ones.

Example 1.8 *A periodic function $f:\mathbb{R}\mapsto\mathbb{R}$ of period 2π is given in the interval $]\pi,\pi]$ by*

$$f(t)=t\,\sin^2 t,\qquad t\in\,]-\pi,\pi].$$

1) Find the Fourier series of the function. Explain why the series is pointwise convergent and find its sum function.

2) Prove that the Fourier series for f is uniformly convergent on $\mathbb{R}$.



1) Clearly, f is piecewise C^1 without vertical half tangents (it is in fact of class C^1; but to prove this will require a fairly long investigation), so $f\in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is pointwise convergent with the sum function $f^*(t)=f(t)$, because $f(t)$ is continuous.

Now, $f(t)$ is *odd*, so $a_n=0$ for every $n\in\mathbb{N}_0$, and

$$\begin{aligned}b_n&=\frac{2}{\pi}\int_0^{\pi}t\,\sin^2 t\,\sin nt\,dt=\frac{1}{\pi}\int_0^{\pi}t(1-\cos 2t)\sin nt\,dt\\&=\frac{1}{2\pi}\int_0^{\pi}t\{2\sin nt-\sin(n+2)t-\sin(n-2)t\}dt.\end{aligned}$$

Then we get for $n\neq 2$ (thus $n-2\neq 0$)

$$\begin{aligned}b_n&=\frac{1}{2\pi}\left[t\left(-\frac{2}{n}\cos nt+\frac{1}{n+2}\cos(n+2)t+\frac{1}{n-2}\cos(n-2)t\right)\right]_0^{\pi}\\&\quad-\frac{1}{2\pi}\int_0^{\pi}\left(-\frac{2}{n}\cos nt+\frac{1}{n+2}\cos(n+2)t+\frac{1}{n-2}\cos(n-2)t\right)dt\\&=\frac{1}{2\pi}\cdot\pi\left\{-\frac{2}{n}(-1)^n+\frac{(-1)^n}{n+2}+\frac{(-1)^n}{n-2}\right\}=\frac{(-1)^n}{2}\left\{\frac{2n}{n^2-4}-\frac{2}{n}\right\}\\&=(-1)^n\left\{\frac{n}{n^2-4}-\frac{1}{n}\right\}=(-1)^n\cdot\frac{4}{n(n^2-4)}.\end{aligned}$$

We get for the exceptional case $n = 2$ that

$$\begin{aligned} b_2 &= \frac{1}{2\pi}\int_0^{\pi} t(2\sin 2t - \sin 4t)\,dt \\ &= \frac{1}{2\pi}\left[t\left(-\cos 2t+\frac{1}{4}\cos 4t\right)\right]_0^{\pi}+\frac{1}{2\pi}\int_0^{\pi}\left(\cos 2t-\frac{1}{4}\cos 4t\right)dt \\ &= \frac{1}{2\pi}\cdot\pi\left(-1+\frac{1}{4}\right)+0=-\frac{3}{8}. \end{aligned}$$

Hence the Fourier series for f is (with pointwise convergence, thus equality sign)

$$f(t) = \frac{4}{3}\sin t - \frac{3}{8}\sin 2t + \sum_{n=3}^{\infty}(-1)^n \cdot \frac{4}{n(n^2-4)}\sin nt.$$

2) Since the Fourier series has the convergent majoring series

$$\frac{4}{3}+\frac{3}{8}+\sum_{n=3}^{\infty}\frac{4}{n(n^2-4)} = \frac{41}{24}+\sum_{n=3}^{\infty}(-1)^n\cdot\frac{4}{(n+1)(n^2+2n-3)} \le \sum_{n=1}^{\infty}\frac{4}{n^3},$$

the Fourier series is uniformly convergent on $\mathbb{R}$.

Example 1.9 *We define an odd function $f \in K_{2\pi}$ by*

$$f(t) = t(\pi - t), \qquad t \in [0,\pi].$$

1) Prove that f has the Fourier series

$$\frac{8}{\pi}\sum_{p=1}^{\infty}\frac{\sin(2p-1)t}{(2p-1)^3}, \qquad t\in\mathbb{R}.$$

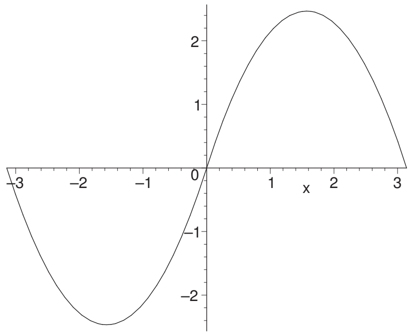
2) Explain why the sum function of the Fourier series is $f(t)$ for every $t \in \mathbb{R}$, and find the sum of the series

$$\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{(2p-1)^3}$$

The graph of the function is an arc of a parabola over $[0,\pi]$ with its vertex at $\left(\frac{\pi}{2},\frac{\pi^2}{4}\right)$. The odd continuation is continuous and piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is *pointwise convergent* with the sum function $f^*(t) = f(t)$.

1) Now, f is odd, so $a_n = 0$. Furthermore, by partial integration,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} t(\pi-t)\sin nt\,dt = -\frac{2}{\pi n}[t(\pi-t)\cos nt]_0^{\pi}+\frac{2}{\pi n}\int_0^{\pi}(\pi-2t)\cos nt\,dt \\ &= 0+\frac{2}{\pi n^2}[(\pi-2t)\sin nt]_0^{\pi}+\frac{4}{\pi n^2}\int_0^{\pi}\sin nt\,dt = 0-\frac{4}{\pi n^3}[\cos nt]_0^{\pi}=\frac{4}{\pi}\cdot\frac{1}{n^3}\{1-(-1)^n\}. \end{aligned}$$



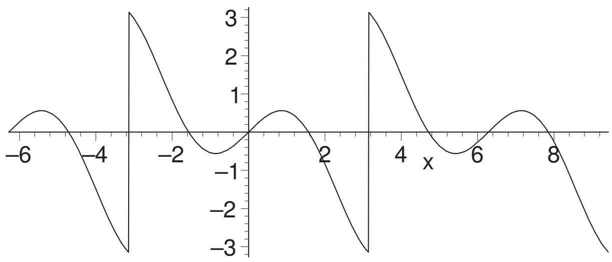
It follows that $b_{2p} = 0$, and that

$$b_{2p-1} = \frac{8}{\pi} \cdot \frac{1}{(2p-1)^3},$$

hence the Fourier series becomes

$$f(t) = \frac{8}{\pi} \sum_{p=1}^{\infty} \frac{\sin(2p-1)t}{(2p-1)^3}$$

where we can use = according to the above.



2) The first question was proved in the beginning of the example.

   If we choose $t = \frac{\pi}{2}$, then

$$f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^3}\sin\left(p\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{(2p-1)^3}.$$

   Then by a rearrangement,

$$\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{(2p-1)^3} = \frac{\pi^3}{32}.$$

Example 1.10 *Let the function $f \in K_{2\pi}$ be given on the interval $]-\pi,\pi]$ by*

$$f(t) = t\cos t.$$

1) Explain why the Fourier series is pointwise convergent in $\mathbb{R}$, and sketch the graph of its sum function in the interval $]-\pi, 3\pi]$.

2) Prove that f has the Fourier series

$$-\frac{1}{2}\sin t + \sum_{n=2}^{\infty}(-1)^n \cdot \frac{2n}{n^2-1}\sin nt, \qquad t \in \mathbb{R}.$$

1) Since f is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. Then by the **main theorem**, the Fourier series is *pointwise convergent everywhere* and its sum function is

$$f^*(t) = \begin{cases} 0 & \text{for } t = \pi + 2p\pi, \quad p \in \mathbb{Z}, \\ f(t) & \text{otherwise.} \end{cases}$$

2) Since $f(t)$ os (almost) odd, we have $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} t\cdot\cos t\cdot\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\,\{\sin(n+1)t + \sin(n-1)t\}\,dt.$$

   For $n = 1$ we get

$$b_1 = \frac{1}{\pi}\int_0^{\pi} t\,\sin 2t\,dt = -\frac{1}{2\pi}[t\,\cos 2t]_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi}\cos 2t\,dt = -\frac{1}{2}.$$

For $n > 1$ we get by partial integration

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[t\left(-\frac{\cos(n+1)t}{n+1}-\frac{\cos(n-1)t}{n-1}\right)\right]_0^\pi + \frac{1}{\pi}\int_0^\pi \left\{\frac{\cos(n+1)t}{n+1}+\frac{\cos(n-1)t}{n-1}\right\} dt \\ &= \frac{1}{\pi}\cdot\pi\left(-\frac{\cos(n+1)\pi}{n+1}-\frac{\cos(n-1)\pi}{n-1}\right)+0=(-1)^n\left(\frac{1}{n+1}+\frac{1}{n-1}\right)=(-1)^n\cdot\frac{2n}{n^2-1}. \end{aligned}$$

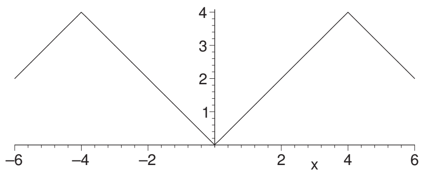
Hence the Fourier series is with pointwise equality

$$f^*(t) = -\frac{1}{2}\sin t + \sum_{n=2}^{\infty}(-1)^n\cdot\frac{2n}{n^2-1}\sin nt.$$

Example 1.11 *A 2π-periodic function is given in the interval $]-\pi,\pi]$ by*

$$f(t) = 2\pi - 3t.$$

1) *Explain why the Fourier series is pointwise convergent for every $t \in \mathbb{R}$, and sketch the graph of its sum function $s(t)$.*
2) *Find the Fourier series for f.*



1) Since f is piecewise C^1 without vertical half tangents, we get $f \in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is pointwise convergent with the sum function

$$s(t) = \begin{cases} 2\pi & \text{for } t = \pi + 2p\pi, \quad p \in \mathbb{Z}, \\ f(t) & \text{otherwise.} \end{cases}$$

The graph of the function $f(t)$ is sketched on the figure.

2) Now, $f(t) = 2\pi - 3t = \frac{1}{2}a_0 - 3t$ is split into its even and its odd part, so it is seen by inspection that $a_0 = 4\pi$, and that the remainder part of the series is a sine series, so $a_n = 0$ for $n \geq 1$, and

$$b_n = \frac{2}{\pi}\int_0^\pi (-3t)\sin nt\,dt = \frac{6}{\pi n}[t\,\cos nt]_0^\pi - \frac{6}{\pi n}\int_0^\pi \cos nt\,dt = (-1)^n\cdot\frac{6}{n},$$

hence (with equality sign instead of $\sim$)

$$s(t) = 2\pi + \sum_{n=1}^{\infty}(-1)^n\cdot\frac{6}{n}\sin nt, \qquad t \in \mathbb{R}.$$

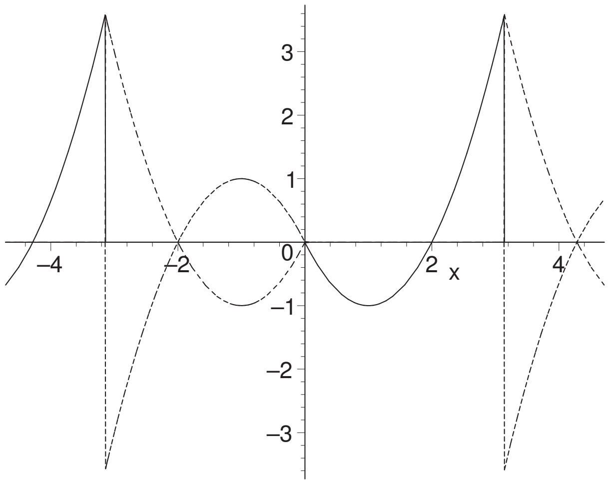
Example 1.12 *Let* $f : [0, \pi] \to \mathbb{R}$ *denote the function given by*

$$f(t) = t^2 - 2t.$$

1) Find the cosine series the sum of which for every $t \in [0, \pi]$ *is equal to* $f(t)$.

2) Find a sine series the sum of which for every $t \in [0, \pi[$ *is equal to* $f(t)$.

Since f is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. The *even* extension is continuous, hence the cosine series is by the **main theorem** *equal to* $f(t)$ in $[0, \pi]$.



The *odd* extension is continuous in the half open interval $[0, \pi[$, hence the **main theorem** only shows that the sum function is $f(t)$ in the half open interval $[0, \pi[$.

1) **Cosine series**. From

$$a_0 = \frac{2}{\pi} \int_0^\pi f(t)\, dt = \frac{2}{\pi} \int_0^\pi (t^2 - 2t) dt = \frac{2}{\pi} \left[\frac{t^3}{3} - t^2 \right]_0^\pi = \frac{2\pi^2}{3} - 2\pi,$$

and for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi (t^2 - 2t) \cos nt\, dt = \frac{2}{\pi n} \left[(t^2 - 2t) \sin nt \right]_0^\pi - \frac{4}{\pi n} \int_0^\pi (t-1) \sin nt\, dt \\ &= 0 + \frac{4}{\pi n^2} [(t-1) \cos nt]_0^\pi - \frac{4}{\pi n^2} \int_0^\pi \cos nt\, dt = \frac{4}{\pi n^2} \{(\pi - 1) \cdot (-1)^n + 1\} + 0, \end{aligned}$$

we get by the initial comments with equality sign

$$f(t) = t^2 - 2t = \frac{\pi^2}{3} - \pi + \sum_{n=1}^\infty \frac{4}{\pi n^2} \{1 + (-1)^n (\pi - 1)\} \cos nt$$

for $t \in [0, \pi]$

2) **Sine series**. Since

$$
\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi}(t^2-2t)\sin nt\,dt = \left[-\frac{2}{\pi n}(t^2-2t)\cos nt\right]_0^{\pi} + \frac{4}{\pi n}\int_0^{\pi}(t-1)\cos nt\,dt \\
&= -\frac{2}{\pi n}\,\pi(\pi-2)\cdot(-1)^n + \frac{4}{\pi n^2}[(t-1)\sin nt]_0^{\pi} - \frac{4}{\pi n^2}\int_0^{\pi}\sin nt\,dt \\
&= \frac{2}{n}(\pi-2)\cdot(-1)^{n-1} + 0 + \frac{4}{\pi n^3}[\cos nt]_0^{\pi} = \frac{2(\pi-2)}{n}\cdot(-1)^{n-1} + \frac{4}{\pi n^3}\left\{(-1)^n-1\right\},
\end{aligned}
$$

we get by the initial comments with equality sign,

$$
f(t)=t^2-2t=\sum_{n=1}^{\infty}\left\{\frac{2(\pi-2)}{n}(-1)^{n-1}-\frac{4}{\pi n^3}\left[1-(-1)^n\right]\right\}\sin nt
$$

for $t \in [0,\pi[$.

Example 1.13 *Find the Fourier series of the periodic function of period 2π, given in the interval $]-\pi,\pi]$ by*

$$f(t)=\begin{cases} t\sin t, & \text{for } t\in[0,\pi],\\ -t\sin t, & \text{for } t\in\,]-\pi,0[,\end{cases}$$

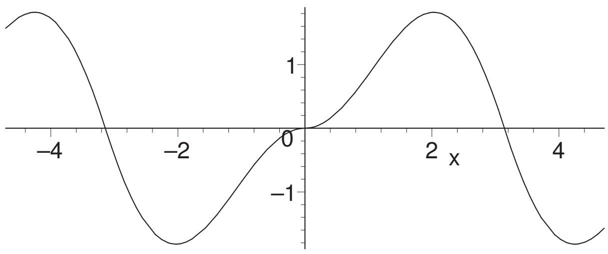
and find for every $t\in\mathbb{R}$ the sum of the series.
Then find for every $t\in[0,\pi]$ the sum of the series

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n+1)^2(2n-1)^2}\cos 2nt.$$

Finally, find the sum of the series

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n+1)^2(2n-1)^2}.$$

Since f is continuous and piecewise C^1 without vertical half tangents, we see that $f\in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is pointwise convergent with the sum $f^*(t)=f(t)$.



Since $f(t)$ is *odd*, the Fourier series is a sine series, hence $a_n=0$, and

$$b_n=\frac{2}{\pi}\int_0^{\pi}t\cdot\sin t\cdot\sin nt\,dt=\frac{1}{\pi}\int_0^{\pi}t\{\cos(n-1)t-\cos(n+1)t\}dt.$$

We get for $n=1$,

$$b_1=\frac{1}{\pi}\int_0^{\pi}t\{1-\cos 2t\}dt=\frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi}-\frac{1}{2\pi}[t\sin 2t]_0^{\pi}+\frac{1}{2\pi}\int_0^{\pi}\sin 2t\,dt=\frac{\pi}{2}.$$

For $n>1$ we get instead,

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[t\left(\frac{\sin(n-1)t}{n-1}-\frac{\sin(n+1)t}{n+1}\right)\right]_0^{\pi}-\frac{1}{\pi}\int_0^{\pi}\left\{\frac{\sin(n-1)t}{n-1}-\frac{\sin(n+1)t}{n+1}\right\}dt\\ &= 0+\frac{1}{\pi}\left[\frac{\cos(n-1)t}{(n-1)^2}-\frac{\cos(n+1)t}{(n+1)^2}\right]_0^{\pi}=\frac{1}{\pi}\left\{\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}\right\}\cdot\left\{(-1)^{n-1}-1\right\}.\end{aligned}$$

It follows that $b_{2n+1}=0$ for $n\geq 1$, and that

$$b_{2n}=-\frac{2}{\pi}\left\{\frac{1}{(2n-1)^2}-\frac{1}{(2n+1)^2}\right\}=-\frac{1}{\pi}\cdot\frac{16n}{(2n-1)^2(2n+1)^2}\qquad\text{for } n\in\mathbb{N}.$$

Hence, the Fourier series is (with an equality sign according to the initial comments)

$$f(t) = \frac{\pi}{2}\,\sin t - \frac{16}{\pi}\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(2n+1)^2}\,\sin 2nt.$$

When we compare with the next question we see that a) we miss a factor n, and b) we have $\sin 2nt$ occurring instead of $\cos 2nt$. However, the formally differentiated series

$$\frac{\pi}{2}\,\cos t - \frac{32}{\pi}\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)^2}\,\cos 2nt$$

has the right structure. Since it has the convergent majoring series

$$\frac{\pi}{2} + \frac{32}{\pi}\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)^2},$$

(the difference between the degree of the denominator and the degree of the numerator is 2, and $\sum n^{-2}$ is convergent), it is absolutely and uniformly convergent, and its derivative is given by

$$f'(t) = \begin{cases} \sin t + t\cos t, & \text{for } t \in]0,\pi[, \\[2mm] -\sin t - t\cos t, & \text{for } t \in]-\pi,0[, \end{cases}$$

where

$$\lim_{t\to 0+} f'(t) = \lim_{t\to 0-} f'(t) = 0 \qquad \text{and} \qquad \lim_{t\to \pi-} f'(t) = \lim_{t\to -\pi+} f'(t) = -\pi$$

The continuation of $f'(t)$ is continuous, hence we conclude that

$$f'(t) = \frac{\pi}{2}\,\cos t - \frac{32}{\pi}\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)^2}\,\cos 2nt,$$

and thus by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)^2}\,\cos 2nt = \frac{\pi^2}{64}\,\cos t - \frac{\pi}{32}\,f'(t) = \frac{\pi^2}{64}\,\cos t - \frac{\pi}{32}\,\sin t - \frac{\pi}{32}\,t\,\cos t \qquad \text{for } t \in [0,\pi].$$

Finally, insert $t=0$, and we get

$$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)^2(2n+1)^2} = \frac{\pi^2}{64}.$$

Alternatively, the latter sum can be calculated by a decomposition and the application of the sum of a known series. In fact, it follows from

$$\begin{aligned} \frac{n^2}{(2n-1)^2(2n+1)^2} &= \frac{1}{16}\cdot\frac{\{(2n-1)+(2n+1)\}^2}{(2n-1)^2(2n+1)^2} = \frac{1}{16}\left\{\frac{(2n+1)^2+(2n-1)^2+2(2n+1)(2n-1)}{(2n-1)^2(2n+1)^2}\right\} \\ &= \frac{1}{16}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}\,\frac{2}{(2n-1)(2n+1)}\right\} = \frac{1}{16}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}+\frac{1}{2n-1}-\frac{1}{2n+1}\right\} \end{aligned}$$

that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)^2(2n+1)^2} &= \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}+\frac{1}{16}\lim_{N\to\infty}\sum_{n=1}^{N}\left\{\frac{1}{2n-1}-\frac{1}{2n+1}\right\}\\ &= \frac{1}{16}\left\{2\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}-1\right\}+\frac{1}{16}\lim_{N\to\infty}\left\{1-\frac{1}{2N+1}\right\}=\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.\end{aligned}$$

Since

$$\begin{aligned}\frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left\{1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right\}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k\\ &= \frac{1}{1-\frac{1}{4}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},\end{aligned}$$

we get

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)^2(2n+1)^2}=\frac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{1}{8}\cdot\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{64}.$$

Example 1.14 *The odd and periodic function f of period 2π, is given in the interval $]0,\pi[$ by*

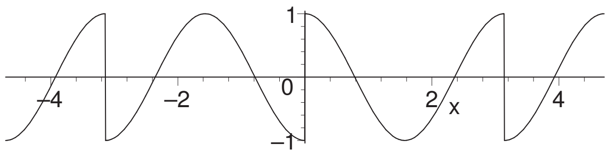
$$f(t) = \cos 2t, \qquad t \in]0,\pi[.$$

1) Find the Fourier series for f.

2) Indicate the sum of the series for $t = \frac{7\pi}{6}$.

3) Find the sum of the series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{2n+1}{(2n-1)(2n+3)}.$$



Since $f(t)$ is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$, so the Fourier series converges according to the **main theorem** pointwise towards the *adjusted* function $f^*(t)$. Since f is *odd*, it is very important to have a figure here. The function $f^*(t)$ is given in $[-\pi,\pi]$ by

$$f^*(t) = \begin{cases} 0 & \text{for } t = -\pi, \\ -\cos 2t & \text{for } t \in]-\pi, 0[, \\ 0 & \text{for } t = 0, \\ \cos 2t & \text{for } t \in]0,\pi[, \\ 0 & \text{for } t = \pi, \end{cases}$$

continued periodically.

1) Now, f is odd, so $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos 2t \cdot \sin nt \, dt = \frac{1}{\pi}\int_0^{\pi} \{\sin(n+2)t + \sin(n-2)t\} dt.$$

Since $\sin(n-2)t = 0$ for $n = 2$, this is the exceptional case. We get for $n = 2$,

$$b_2 = \frac{1}{\pi}\int_0^{\pi} \sin 4t \, dt = \frac{1}{\pi}\left[-\frac{\cos 4t}{4}\right]_0^{\pi} = 0.$$

Then for $n \neq 2$,

$$b_n = \frac{1}{\pi}\left[-\frac{\cos(n+2)t}{n+2} - \frac{\cos(n-2)t}{n-2}\right]_0^{\pi} = -\frac{1}{\pi}\left(\frac{1}{n+2} + \frac{1}{n-2}\right)\{(-1)^n - 1\}.$$

It follows that $b_{2n} = 0$ for $n > 1$ (and also for $n = 1$, by the earlier investigation of the exceptional case), and that

$$b_{2n+1} = -\frac{1}{\pi}\left(\frac{1}{2n+3} + \frac{1}{2n-1}\right) \cdot (-2) = \frac{4}{\pi} \cdot \frac{2n+1}{(2n-1)(2n+3)}.$$

Summing up we get the Fourier series (with an equality sign instead of the difficult one, $\sim$)

$$(2)\quad f^*(t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{2n+1}{(2n-1)(2n+3)} \sin(2n+1)t.$$

2) This question is very underhand, cf. the figure. It follows from the periodicity that the sum of the series for $t = \dfrac{7\pi}{6} > \pi$, is given by

$$f\left(\frac{7\pi}{6}\right) = f\left(\frac{7\pi}{6} - 2\pi\right) = f\left(-\frac{5\pi}{6}\right) = -\cos\left(-\frac{5\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}.$$

3) The coefficient of the series is the same as in the Fourier series, so we shall only choose t in such a way that $\sin(2n+1)t$ becomes equal to ± 1.

We get for $t = \dfrac{\pi}{2}$,

$$\sin(2n+1)\frac{\pi}{2} = \sin n\pi \cdot \cos\frac{\pi}{2} + \cos n\pi \cdot \sin\frac{\pi}{2} = (-1)^n,$$

hence by insertion into (2),

$$f^*\left(\frac{\pi}{2}\right) = \cos\pi = -1 = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{2n+1}{(2n-1)(2n+3)}(-1)^n,$$

and finally by a rearrangement,

$$\sum_{n=0}^{\infty}(-1)^{n+1} \cdot \frac{2n+1}{(2n-1)(2n+3)} = \frac{\pi}{4}.$$

Remark 1.2 The last question can also be calculated by means of a decomposition and a consideration of the sectional sequence (an Arctan series). The sketch of this alternative proof is the following,

$$\sum_{n=0}^{\infty}(-1)^{n+1} \cdot \frac{2n+1}{(2n-1)(2n+3)} = \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \text{Arctan } 1 = \frac{\pi}{4}.$$

The details, i.e. the dots, are left to the reader.

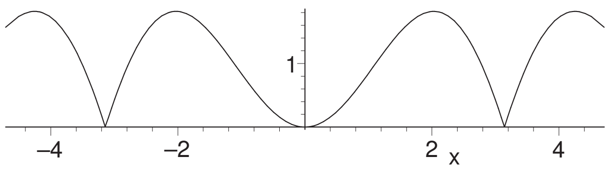
Example 1.15 *Find the Fourier series the function $f \in K_{2\pi}$, which is given in the interval $[-\pi, \pi]$ by*

$$f(t) = t \cdot \sin t.$$

Find by means of this Fourier series the sum function of the trigonometric series

$$\sum_{n=2}^{\infty} \frac{(-1)^n \sin nt}{(n-1)n(n+1)} \qquad \text{for } t \in [-\pi, \pi].$$

Since f is continuous and piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. Then by the **main theorem**, the Fourier series is pointwise convergent everywhere and its sum function is $f^*(t) = f(t)$.



Since f is *even*, the Fourier series is a cosine series, thus $b_n = 0$, and

$$a_n = \frac{2}{\pi} \int_0^{\pi} t \sin t \cos nt \, dt = \frac{1}{\pi} \int_0^{\pi} t\{\sin(n+1)t - \sin(n-1)t\} dt.$$

The exceptional case is $n = 1$, in which $\sin(n-1)t = 0$ identically. For $n = 1$ we calculate instead,

$$a_1 = \frac{1}{\pi} \int_0^{\pi} t \sin 2t \, dt = \frac{1}{2\pi}[-t \cos 2t]_0^{\pi} + \frac{1}{2\pi} \int_0^{\pi} \cos 2t \, dt = \frac{-\pi}{2\pi} = -\frac{1}{2}.$$

For $n \neq 1$ we get

$$\begin{aligned} a_n &= \frac{1}{\pi} \left[t \left(-\frac{\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1} \right) \right]_0^{\pi} + \frac{1}{\pi} \int_0^{\pi} \left\{ \frac{\cos(n+1)t}{n+1} - \frac{\cos(n-1)t}{n-1} \right\} dt \\ &= \frac{1}{\pi} \cdot \pi \left(-\frac{1}{n+1} + \frac{1}{n-1} \right) \cdot (-1)^{n+1} = (-1)^{n+1} \cdot \frac{2}{(n-1)(n+1)}. \end{aligned}$$

According to the initial remarks we get with pointwise equality sign,

$$f(t) = t \sin t = 1 - \frac{1}{2} \cos t + 2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)(n+1)} \cos nt, \quad \text{for } t \in [-\pi, \pi].$$

The Fourier series has the convergent majoring series

$$1 + \frac{1}{2} + 2 \sum_{n=2}^{\infty} \frac{1}{n^2 - 1},$$

hence it is uniformly convergent. We may therefore integrate it term by term,

$$\int_0^t f(\tau)\,d\tau = t - \frac{1}{2}\,\sin t + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)n(n+1)}\,\sin nt,$$

for $t \in [-\pi, \pi]$.

Hence by a rearrangement for $t \in [-\pi, \pi]$,

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{(-1)^n \sin nt}{(n-1)n(n+1)} &= \frac{1}{2}\,t - \frac{1}{4}\,\sin t - \frac{1}{2}\int_0^t f(\tau)\,d\tau = \frac{1}{2}\,t - \frac{1}{4}\,\sin t - \frac{1}{2}\int_0^t \tau\,\sin\tau\,d\tau \\
&= \frac{1}{2}\,t - \frac{1}{4}\,\sin t - \frac{1}{2}[-\tau\,\cos\tau + \sin\tau]_0^t = \frac{1}{2}\,t - \frac{1}{4}\,\sin t + \frac{1}{2}\,t\,\cos t - \frac{1}{2}\,\sin t \\
&= \frac{1}{2}\,t + \frac{1}{2}\,t\,\cos t - \frac{3}{4}\,\sin t = \frac{1}{2}\,t(1+\cos t) - \frac{3}{4}\,\sin t.
\end{aligned}$$

Example 1.16 *Prove that for every* $n \in \mathbb{N}$,

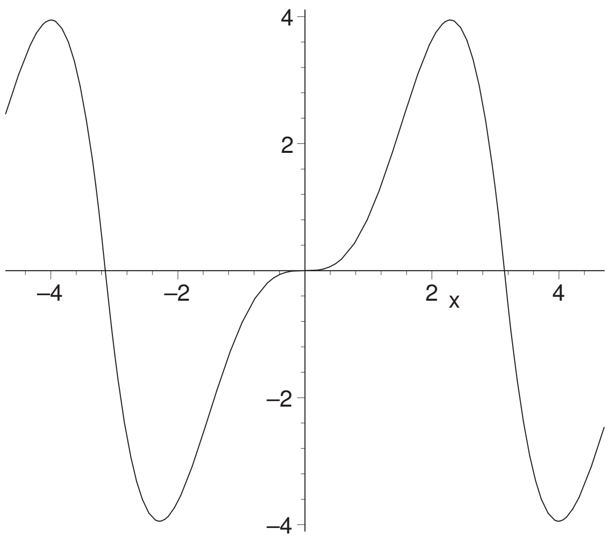
$$\int_0^\pi t^2 \cos nt\, dt = (-1)^n \cdot \frac{2\pi}{n^2}.$$

Find the Fourier series for the function $f \in K_{2\pi}$, *given in the interval* $[-\pi, \pi]$ *by*

$$f(t) = t^2 \sin t.$$

Then write the derivative $f'(t)$ *by means of a trigonometric series and find the sum of the series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(2n)^2}{(2n-1)^2(2n+1)^2}.$$



We get by partial integration,

$$\int_0^\pi t^2 \cos nt\, dt = \left[\frac{1}{n}\, t^2 \sin nt\right]_0^\pi - \frac{2}{n}\int_0^\pi t\,\sin nt\, dt = 0 + \left[\frac{2t}{n^2}\,\cos nt\right]_0^\pi - \frac{2}{n^2}\int_0^\pi \cos nt\, dt = (-1)^n \cdot \frac{2\pi}{n^2}.$$

The function f is continuous and piecewise C^1 without vertical half tangents, hence $f \in K_{2\pi}^*$. By the **main theorem** the Fourier series is pointwise convergent everywhere and its sum function is $f^*(t) = f(t)$.

Since f is *odd*, its Fourier series is a sine series, thus $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^\pi t^2 \sin t\, \sin nt\, dt = \frac{1}{\pi}\int_0^\pi t^2\{\cos(n-1)t - \cos(n+1)t\} dt.$$

For $n = 1$ we get by the result above,

$$b_1 = \frac{1}{\pi}\int_0^\pi t^2(1 - \cos 2t) dt = \frac{1}{\pi} \cdot \frac{\pi^3}{3} - \frac{1}{\pi} \cdot (-1)^2 \cdot \frac{2\pi}{4} = \frac{\pi^2}{3} - \frac{1}{2}.$$

For $n>1$ we also get by the result above,

$$\begin{aligned} b_n &= \frac{1}{\pi}\cdot 2\pi\left\{\frac{(-1)^{n-1}}{(n-1)^2}-\frac{(-1)^{n+1}}{(n+1)^2}\right\}=(-1)^{n-1}\cdot 2\cdot\frac{(n+1)^2-(n-1)^2}{(n-1)^2(n+1)^2}\\ &= (-1)^{n-1}\cdot\frac{8n}{(n-1)^2(n+1)^2}. \end{aligned}$$

According to the initial comments we have equality sign for $t\in[-\pi,\pi]$,

$$f(t)=t^2\sin t=\left(\frac{\pi^2}{3}-\frac{1}{2}\right)\sin t+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot 8n}{(n-1)^2(n+1)^2}\,\sin nt.$$

By a formal termwise differentiation of the Fourier series we get

$$\left(\frac{\pi^2}{3}-\frac{1}{2}\right)\cos t+8\sum_{n=2}^{\infty}(-1)^{n-1}\cdot\frac{n^2}{(n-1)^2(n+1)^2}\,\cos nt.$$

This has the convergent majoring series

$$\frac{\pi^2}{3}-\frac{1}{2}+8\sum_{n=2}^{\infty}\frac{n^2}{(n-1)^2(n+1)^2},$$

hence it is uniformly convergent and its sum function is

$$f'(t)=t^2\cos t+2t\,\sin t=\left(\frac{\pi^2}{3}-\frac{1}{2}\right)\cos t+8\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot n^2}{(n-1)^2(n+1)^2}\,\cos nt.$$

When we insert $t=\dfrac{\pi}{2}$, we get

$$\begin{aligned} f'\left(\frac{\pi}{2}\right) &= 0+\pi=\pi=0+8\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cdot n^2}{(n-1)^2(n+1)^2}\,\cos\left(n\,\frac{\pi}{2}\right)\\ &= 8\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}\cdot(2n)^2}{(2n-1)^2(2n+1)^2}\,\cos(n\pi)+0\\ &= 8\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{(2n)^2}{(2n-1)^2(2n+1)^2}, \end{aligned}$$

hence by a rearrangement,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{(2n)^2}{(2n-1)^2(2n+1)^2}=\frac{\pi}{8}.$$

Alternatively, we get by a decomposition,

$$\frac{(2n)^2}{(2n-1)^2(2n+1)^2}=\frac{1}{4}\left\{\frac{1}{(2n-1)^2}+\frac{1}{(2n+1)^2}+\frac{1}{2n-1}-\frac{1}{2n+1}\right\},$$

thus

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n)^2}{(2n-1)^2}$$
$$=\frac{1}{4}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)^2}+\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right\}$$
$$=\frac{1}{4}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}-\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\right\}+\frac{1}{4}\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}+\sum_{n=2}^{N+1}\frac{(-1)^{n-1}}{2n-1}\right\}$$
$$=\frac{1}{4}+\frac{1}{4}\lim_{N\to\infty}\left\{2\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}-1+\frac{(-1)^N}{2N+1}\right\}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}=\frac{1}{2}\,\text{Arctan}\,1=\frac{\pi}{8}.$$

Example 1.17 *The odd and periodic function f of period 2π is given in the interval $[0,\pi]$ by*

$$f(t)=\begin{cases}\sin t, & \textit{for } t\in\left[0,\dfrac{\pi}{2}\right],\\[2mm] -\sin t, & \textit{for } t\in\left]\dfrac{\pi}{2},\pi\right]\end{cases}$$

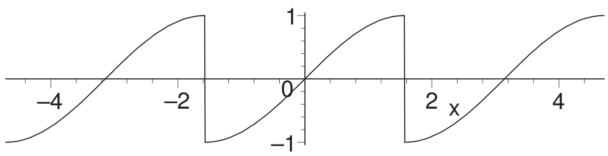
1) Find the Fourier series of the function. Explain why the series is pointwise convergent, and find its sum for every $t\in[0,\pi]$.

2) Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(4n+1)(4n+3)}.$$

Since f is piecewise C^1 without vertical half tangents, we have $f\in K_{2\pi}^*$. By the **main theorem** the Fourier series is pointwise convergent and its sum is

$$f^*(t)=\begin{cases}0 & \text{for } t=\dfrac{\pi}{2}+p\pi,\ p\in\mathbb{Z},\\ f(t) & \text{otherwise.}\end{cases}$$



1) Since f is *odd*, we have $a_n = 0$, and for $n > 1$ we get

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi/2} \sin t\,\sin nt\,dt - \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin t\,\sin nt\,dt \\
&= \frac{1}{\pi}\int_0^{\pi/2}\{\cos(n-1)t - \cos(n+1)t\}dt - \frac{1}{\pi}\int_{\pi/2}^{\pi}\{\cos(n-1)t - \cos(n+1)t\}dt \\
&= \frac{1}{\pi}\left\{\left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_0^{\pi/2} + \left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_{\pi}^{\pi/2}\right\} \\
&= \frac{2}{\pi}\left\{\frac{\sin(n-1)\frac{\pi}{2}}{n-1} - \frac{\sin(n+1)\frac{\pi}{2}}{n+1}\right\}.
\end{aligned}$$

Hence $b_{n+1} = 0$ for $n \geq 1$, and

$$\begin{aligned} b_{2n} &= \frac{2}{\pi}\left\{\frac{\sin\left(n\pi-\dfrac{\pi}{2}\right)}{2n-1} - \frac{\sin\left(n\pi+\dfrac{\pi}{2}\right)}{2n+1}\right\} = \frac{2}{\pi}(-1)^{n-1}\left\{\frac{1}{2n-1}+\frac{1}{2n+1}\right\} \\ &= (-1)^{n-1}\cdot\frac{2}{\pi}\cdot\frac{4n}{(2n-1)(2n+1)}, \qquad n \in \mathbb{N}. \end{aligned}$$

For $n = 1$ (the exceptional case) we get

$$b_1 = \frac{2}{\pi}\left\{\int_0^{\pi/2}\sin^2 t\,dt - \int_{\pi/2}^{\pi}\sin^2 t\,dt\right\} = \frac{2}{\pi}\left\{\int_0^{\pi/2}\sin^2 t\,dt - \int_0^{\pi/2}\sin^2 t\,dt\right\} = 0.$$

Summing up we get the Fourier series

$$f \sim \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{8}{\pi}\cdot\frac{n}{(2n-1)(2n+1)}\,\sin 2nt.$$

The sum is in $[0,\pi]$ given by

$$\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,n}{(2n-1)(2n+1)} = \begin{cases} \sin t & \text{for } t \in \left[0,\dfrac{\pi}{2}\right[, \\[2mm] 0 & \text{for } t = \dfrac{\pi}{2}, \\[2mm] -\sin t & \text{for } t \in \left]\dfrac{\pi}{2},\pi\right]. \end{cases}$$

2) When we put $t = \dfrac{\pi}{4}$ into the Fourier series, we get

$$\begin{aligned} \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} &= \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{(2n-1)(2n+1)}\,\sin\left(n\,\frac{\pi}{2}\right) \\ &= \frac{8}{\pi}\sum_{p=0}^{\infty}(-1)^{2p+1-1}\cdot\frac{2p+1}{(4p+1)(4p+3)}\cdot(-1)^p, \end{aligned}$$

hence

$$\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2n+1}{(4n+1)(4n+3)} = \frac{\pi\sqrt{2}}{8}.$$

Example 1.18 *1) Given the infinite series*

$$a)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1},\quad b)\ \sum_{n=1}^{\infty}\frac{n}{n^2+1},\quad c)\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2n-1)}{(2n-1)^2+1}.$$

Explain why the series of a) and c) are convergent, while the series of b) is divergent.

2) Prove for the series of a) that the difference between its sum s and its n-th member of its sectional sequence s_n is numerically smaller than 10^{-1}, when $n \geq 9$.

3) Let a function $f \in K_{2\pi}$ be given by

$$f(t) = \sinh t \qquad \text{for } -\pi < t \leq \pi.$$

Prove that the Fourier series for f is

$$\frac{2\sinh\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2+1}\,\sin nt.$$

4) Find by means of the result of (3) the sum of the series c) in (1).

1) Since $\dfrac{n}{n^2+1} \sim \dfrac{1}{n}$, and $\sum_{n=1}^{\infty}\dfrac{1}{n}$ is divergent, it follows from the **criterion of equivalence** that b) is divergent. It also follows that neither a) nor c) can be absolutely convergent. Since $a_n \to 0$ for $n \to \infty$, we must apply **Leibniz's criterion**. Clearly, both series are *alternating*. If we put

$$\varphi(x) = \frac{x}{x^2+1}, \qquad \text{er} \qquad \varphi'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(1+x^2)^2} < 0$$

for $x > 1$, then $\varphi(x) \to 0$ *decreasingly* for $x \to \infty$, $x > 1$. Then it follows from **Leibniz's criterion** that both a) and c) are (conditionally) convergent.

2) Since a) is alternating, the error is at most equal to the first neglected term, hence

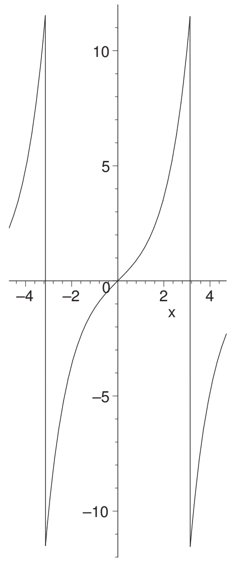
$$|s - s_n| \leq |s - s_9| \leq |a_{10}| = \frac{10}{10^2+1} = \frac{10}{101} < \frac{1}{10} \quad \text{for } n \geq 9.$$

3) Since f is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. Then by the **main theorem**, the Fourier series is pointwise convergent and its sum function is

$$f^*(t) = \begin{cases} 0 & \text{for } t = (2p+1)\pi,\ p \in \mathbb{Z} \\ f(t) & \text{ellers.} \end{cases}$$

Since f is *odd*, we have $a_n = 0$, and

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi}\sinh t\cdot\sin nt\,dt = -\frac{2}{\pi n}[\sinh t\cdot\cos nt]_0^{\pi} + \frac{2}{\pi n}\int_0^{\pi}\cosh t\cdot\cos nt\,dt \\ &= \frac{2}{\pi n}\sinh\pi\cdot(-1)^{n+1} + \frac{2}{\pi n^2}[\cosh t\cdot\sin nt]_0^{\pi} - \frac{2}{\pi n^2}\int_0^{\pi}\sinh t\cdot\sin nt\,dt \\ &= (-1)^{n+1}\cdot\frac{2}{n}\cdot\frac{\sinh\pi}{\pi} - \frac{1}{n^2}\,b_n, \end{aligned}$$



hence by a rearrangement,

$$b_n = \left(1+\frac{1}{n^2}\right)^{-1} \cdot (-1)^{n+1} \cdot \frac{2}{n} \cdot \frac{\sinh \pi}{\pi} = \frac{2\sinh \pi}{\pi} \cdot \frac{(-1)^{n+1} n}{n^2+1}.$$

The Fourier series is (with equality sign, cf. the above)

$$f^*(t) = \sinh t = \frac{2\sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} \sin nt \text{ for } t \in]-\pi, \pi[.$$

4) When we put $t = \frac{\pi}{2}$ into the Fourier series, we get

$$\begin{aligned} \sinh \frac{\pi}{2} &= 2\frac{\sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1} \sin\left(n\,\frac{\pi}{2}\right) = 2\,\frac{\sinh \pi}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^{2p}(2p-1)}{(2p-1)^2+1} \sin\left(p\pi - \frac{\pi}{2}\right) \\ &= 4\,\frac{\sinh \frac{\pi}{2}\, \cosh \frac{\pi}{2}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n-1)^2+1}, \end{aligned}$$

hence by a rearrangement

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-1)}{(2n-1)^2+1} = \frac{\pi}{4\cosh \frac{\pi}{2}}.$$

Example 1.19 *The even and periodic function f of period 2π is given in the interval $[0,\pi]$ by*

$$f(x)=\begin{cases} k-k^2x, & x\in\left[0,\dfrac{1}{k}\right],\\[2ex] 0, & x\in\left]\dfrac{1}{k},\pi\right],\end{cases}$$

where $k\in\left]\dfrac{1}{\pi},\infty\right[$.

1) *Find the Fourier series of the function. Explain why the series is uniformly convergent, and find its sum for* $x=\dfrac{1}{k}$.

2) *Explain why the series*

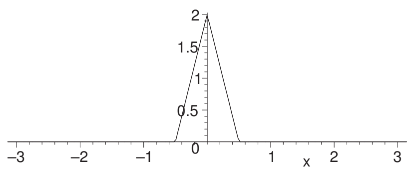
$$\sum_{n=1}^{\infty}\frac{\cos n}{n^2}\quad\textit{and}\quad\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}$$

are convergent, and prove by means of (1) that

$$\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^2}=\frac{1}{4}+\sum_{n=1}^{\infty}\frac{\cos n}{n^2}.$$

3) *In the Fourier series for f we denote the coefficient of $\cos nx$ by $a_n(k)$, $n\in\mathbb{N}$. Prove that* $\lim_{k\to\infty}a_n(k)$ *exists for every $n\in\mathbb{N}$ and that it does not depend on n.*

1) It follows by a consideration of the figure that $f\in K_{2\pi}^*$ and that f is continuous. Then by the **main theorem**, f is the sum function for its Fourier series.



Since f is *even*, we get $b_n=0$, and for $n\in\mathbb{N}$ we find

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{1/k}(k-k^2x)\cos nx\,dx=\frac{2}{\pi n}\left[(k-k^2x)\sin nx\right]_0^{1/k}+\frac{2k^2}{\pi n}\int_0^{1/k}\sin nx\,dx\\ &= \frac{2k^2}{\pi n^2}\left\{1-\cos\left(\frac{n}{k}\right)\right\}.\end{aligned}$$

Since

$$a_0=\frac{2}{\pi}\int_0^{1/k}(k-k^2x)dx=\frac{2}{\pi}\left[kx-\frac{1}{2}\,k^2x^2\right]_0^{1/k}=\frac{1}{\pi},$$

the Fourier series becomes (with equality sign, cf. the above)

$$f(x) = \frac{1}{2\pi} + \frac{2k^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{1 - \cos\left(\frac{n}{k}\right)\right\} \cos nx.$$

Since $\frac{1}{2\pi} + \frac{2k^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent majoring series, the Fourier series is uniformly convergent.

When $x = \frac{1}{k}$, the sum is equal to

$$(3) \quad f\left(\frac{1}{k}\right) = 0 = \frac{1}{2\pi} + \frac{2k^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left\{1 - \cos\frac{n}{k}\right\} \cos\frac{n}{k}.$$

2) Since $\frac{2k^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent majoring series, the series of (3) can be split. Then by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos^2\left(\frac{n}{k}\right) = \frac{1}{4k^2} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\left(\frac{n}{k}\right) \qquad \text{for every } k > \frac{1}{\pi}.$$

If we especially choose $k = 1 > \frac{1}{\pi}$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos^2 n = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n.$$

3) Clearly,

$$a_0(k) = \frac{1}{\pi} \to \frac{1}{\pi} \qquad \text{for } k \to \infty.$$

For $n > 0$ it follows by a Taylor expansion,

$$\begin{aligned} a_n(k) &= \frac{2k^2}{\pi n^2} \left\{1 - \cos\frac{n}{k}\right\} = \frac{2k^2}{\pi n^2} \left\{1 - \left(1 - \frac{1}{2}\,\frac{n^2}{k^2} + \frac{n^2}{k^2}\varepsilon\left(\frac{n}{k}\right)\right)\right\} \\ &= \frac{1}{\pi} + \varepsilon\left(\frac{n}{k}\right) \to \frac{1}{\pi} \qquad \text{for } k \to \infty. \end{aligned}$$

Example 1.20 *Given the function $f \in K_{2\pi}$, where*

$$f(t) = \cos \frac{t}{2}, \qquad -\pi < t \leq \pi.$$

1) Sketch the graph of f.

2) Prove that f has the Fourier series

$$\frac{2}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}} \cos nt,$$

and explain why the Fourier series converges pointwise towards f on $\mathbb{R}$.

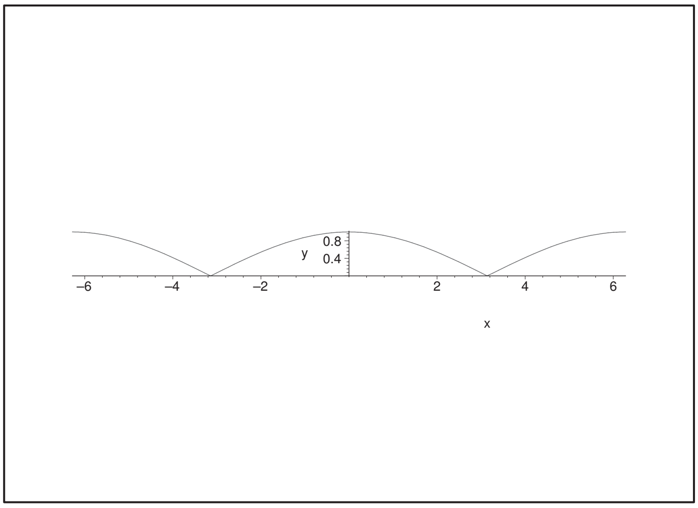
3) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}}.$$

1) Clearly, f is piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$, and we can apply the **main theorem**. Now, $f(t)$ is continuous, hence the adjusted function is $f(t)$ itself, and we have with an equality sign,

$$f(t) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

where we have used that $f(t)$ is even, so $b_n = 0$. We have thus proved (1) and the latter half of (2).



2) **Calculation of the Fourier coefficients**. It follows from the above that $b_n = 0$. Furthermore,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos\frac{t}{2}\,dt = \frac{4}{\pi}\left[\sin\frac{t}{2}\right]_0^{\pi} = \frac{4}{\pi},$$

and

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}\cos\frac{t}{2}\,\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}\left\{\cos\left(n+\frac{1}{2}\right)t+\cos\left(n-\frac{1}{2}\right)t\right\}dt \\
&= \frac{1}{\pi}\left[\frac{1}{n+\frac{1}{2}}\,\sin\left(n+\frac{1}{2}\right)t+\frac{1}{n-\frac{1}{2}}\,\sin\left(n-\frac{1}{2}\right)t\right]_0^{\pi} \\
&= \frac{1}{\pi}\left\{\frac{(-1)^n}{n+\frac{1}{2}}-\frac{(-1)^n}{n-\frac{1}{2}}\right\} = \frac{(-1)^n}{\pi}\cdot\frac{\left(n-\frac{1}{2}\right)-\left(n+\frac{1}{2}\right)}{n^2-\frac{1}{4}} = \frac{(-1)^{n+1}}{\pi}\cdot\frac{1}{n^2-\dfrac{1}{4}}.
\end{aligned}$$

Hence, the Fourier series is (with equality, cf. (1))

$$f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nt = \frac{2}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2-\dfrac{1}{4}}\,\cos nt.$$

Alternative proof of the convergence. Since the Fourier series has the convergent majoring series

$$\frac{2}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}},$$

it is uniformly convergent, hence also pointwise convergent.

3) The sum function is $f(t)$, hence for $t = 0$,

$$f(0) = \cos 0 = 1 = \frac{2}{\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}},$$

and we get by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - \frac{1}{4}} = \pi - 2.$$

Example 1.21 *The even and periodic function f of period 2π ia given in the interval $[0, \pi]$ by*

$$\begin{cases} (t - (\pi/2))^2, & t \in [0, \pi/2], \\ 0, & t \in]\pi/2, \pi]. \end{cases}$$

1) Sketch the graph of f in the interval $[-\pi, \pi]$ and explain why f is everywhere pointwise equal its Fourier series.

2) Prove that

$$f(t) = \frac{\pi^2}{24} + 2 \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{2}{\pi n^3} \sin n\frac{\pi}{2} \right) \cos nt, \quad t \in \mathbb{R}.$$

3) Find by using the result of (2) the sum of the series

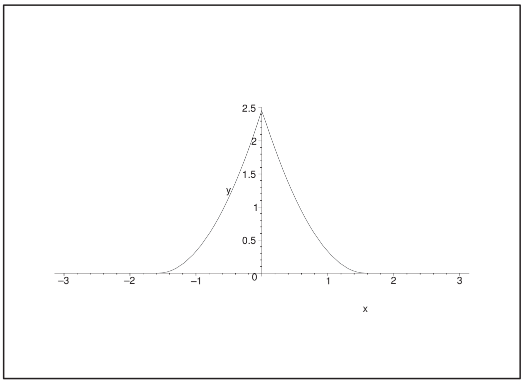
$$\sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{p^2}. \qquad \left(\textit{Hint: Insert } t = \frac{\pi}{2} \right).$$

1) Since f is piecewise C^1 without vertical half tangents, we see that $f \in K_{2\pi}^*$. Since f is continuous, we have $f^* = f$. Since f is even, it follows that $b_n = 0$, hence we have with equality sign by the **main theorem** that

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt = \frac{2}{\pi} \int_0^{\pi/2} \left(t - \frac{\pi}{2} \right)^2 \cos nt \, dt, \quad n \in \mathbb{N}_0.$$



2) We have $b_n = 0$ (an even function), and

$$a_0 = \frac{2}{\pi}\int_0^{\pi/2}\left(t-\frac{\pi}{2}\right)^2 dt = \frac{2}{\pi}\left[\frac{1}{3}\left(t-\frac{\pi}{2}\right)^3\right]_0^{\pi/2} = \frac{\pi^2}{12}.$$

For $n \in \mathbb{N}$ we get by partial integration

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi/2}\left(t-\frac{\pi}{2}\right)^2 \cos nt\,dt = \frac{2}{\pi}\left[\left(t-\frac{\pi}{2}\right)^2\cdot\frac{\sin nt}{n}\right]_0^{\pi/2} - \frac{4}{\pi n}\int_0^{\pi/2}\left(t-\frac{\pi}{2}\right)\sin nt\,dt \\ &= 0+\frac{4}{\pi n}\left[\left(t-\frac{\pi}{2}\right)\cdot\frac{\cos nt}{n}\right]_0^{\pi/2} - \frac{4}{\pi n^2}\int_0^{\pi/2}\cos nt\,dt \\ &= -\frac{4}{\pi n}\left(-\frac{\pi}{2}\right)\cdot\frac{1}{n} - \frac{4}{\pi n^2}\left[\frac{\sin nt}{n}\right]_0^{\pi/2} = \frac{2}{n^2} - \frac{4}{\pi n^3}\sin n\frac{\pi}{2}. \end{aligned}$$

Hence the Fourier series is

$$(4)\quad f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nt = \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{2}{\pi n^3}\sin n\frac{\pi}{2}\right)\cos nt, \quad t \in \mathbb{R}.$$

3) When we insert $t = \frac{\pi}{2}$ into (4) we get

$$
\begin{aligned}
f\left(\frac{\pi}{2}\right) = 0 \quad &= \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{2}{\pi n^3}\sin n\frac{\pi}{n}\right)\cos n\frac{\pi}{2} \\
&= \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos n\frac{\pi}{2} - \frac{2\sin n\frac{\pi}{2}\cos n\frac{\pi}{2}}{\pi n^3}\right) \\
&= \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos n\frac{\pi}{2} - \frac{1}{\pi n^3}\sin n\pi\right) \\
&= \frac{\pi^2}{24} + 2\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\frac{\pi}{2} \\
&= \frac{\pi^2}{24} + 2\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\cos\left(\frac{\pi}{2}+p\pi\right) + 2\sum_{p=1}^{\infty}\frac{1}{(2p)^2}\cos p\pi \\
&= \frac{\pi^2}{24} + \frac{1}{2}\sum_{p=1}^{\infty}\frac{(-1)^p}{p^2},
\end{aligned}
$$

because $\cos\left(\frac{\pi}{2}+p\pi\right)=0$ for $p\in\mathbb{Z}$.

Then by a rearrangement,

$$\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{p^2}=\frac{\pi^2}{12}.$$

Example 1.22 *An even function $f\in K_{4\ell}$ is given in the interval $[0,2\ell]$ by*

$$f(t)=\begin{cases}1 & \text{for } 0\le t\le \ell/2,\\[2mm] 1/2 & \text{for } \ell/2<t\le 3\ell/2,\\[2mm] 0 & \text{for } 3\ell/2<t\le 2\ell.\end{cases}$$

1) Sketch the graph of f in the interval $-3\ell\le t\le 3\ell$, and find the angular frequency ω.
When we answer the next question, the formula at the end of this example may be helpful.

2) a) Give reasons for why the Fourier series for f is of the form

$$f\sim\frac{1}{2}\,a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{2\ell}\right),$$

and find the value of a_0.

b) Prove that $a_n=0$ for $n=2,\,4,\,6,\,\cdots$.

c) Prove that for n odd a_n may be written as

$$a_n=\frac{2}{n\pi}\,\sin\left(n\,\frac{\pi}{4}\right).$$

3) It follows from the above that

$$f\sim\frac{1}{2}+\frac{\sqrt{2}}{\pi}\left\{\cos\frac{\pi t}{2\ell}+\frac{1}{3}\cos\frac{3\pi t}{2\ell}-\frac{1}{5}\cos\frac{5\pi t}{2\ell}-\frac{1}{7}\cos\frac{7\pi t}{2\ell}+\cdots\right\}.$$

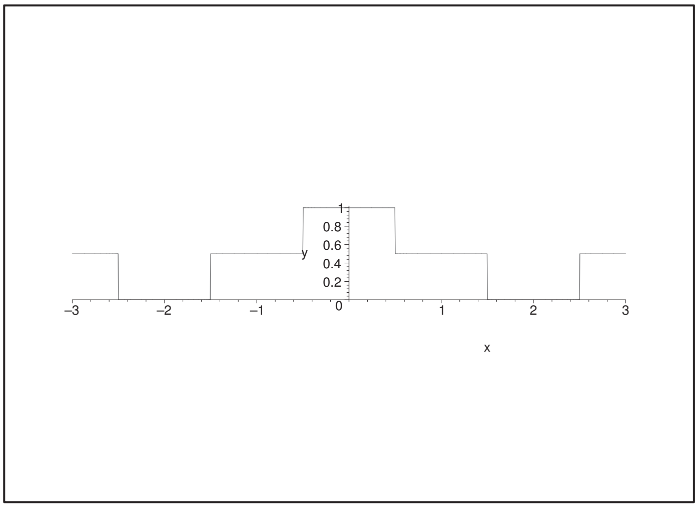
Apply the theory of Fourier series to find the sum of the following two series,

$$(1)\;\; 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\cdots,$$

$$(2)\;\; 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\cdots.$$

The formula to be used in (2):

$$\sin u+\sin v=2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right).$$



1) The angular frequency is $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{4\ell} = \dfrac{\pi}{2\ell}$.

   Since f is piecewise constant, f is piecewise C^1 without vertical half tangents, thus $f \in K_{4\ell}^*$. According to the **main theorem**, the Fourier series is pointwise convergent everywhere with the adjusted function $f^*(t)$ as its sum function. Here $f^*(t) = f(t)$, with the exception of the discontinuities of f, in which the value is the mean value.

2) a) Since f is even and $\omega = \dfrac{\pi}{2\ell}$, the Fourier series has the structure

$$f \sim \frac{1}{2}\,a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi t}{2\ell} = f^*(t),$$

   where

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt = \frac{1}{\ell}\int_0^{2\ell} f(t)\,dt = \frac{1}{\ell}\left\{1\cdot\frac{\ell}{2} + \frac{1}{2}\cdot\ell\right\} = 1.$$

   b) If we put $n = 2p$, $p \in \mathbb{N}$, then

$$\begin{aligned}
a_{2p} &= \frac{1}{\ell}\int_0^{2\ell} f(t)\cos\frac{2p\pi t}{2\ell}\,dt = \frac{1}{\ell}\int_0^{2\ell} f(t)\cos\frac{p\pi t}{\ell}\,dt \\
&= \frac{1}{\ell}\left\{1\cdot\int_0^{\ell/2}\cos\frac{p\pi t}{\ell}\,dt + \frac{1}{2}\int_{\ell/2}^{3\ell/2}\cos\frac{p\pi t}{\ell}\,dt\right\} \\
&= \frac{1}{\ell}\left\{\frac{\ell}{p\pi}\left[\sin\frac{p\pi t}{\ell}\right]_{t=0}^{\ell/2} + \frac{1}{2}\cdot\frac{\ell}{p\pi}\left[\sin\frac{p\pi t}{\ell}\right]_{\ell/2}^{3\ell/2}\right\} \\
&= \frac{1}{2p\pi}\left\{\sin\frac{p\pi}{2} + \sin p\cdot\frac{3\pi}{2}\right\} = \frac{1}{2p\pi}\cdot 2\sin(p\pi)\cdot\cos p\cdot\frac{\pi}{2} = 0.
\end{aligned}$$

c) If instead $n = 2p+1$, $p \in \mathbb{N}_0$, then

$$\begin{aligned} a_{2p+1} &= \frac{1}{\ell}\left\{\int_0^{\ell/2} f(t)\cos\frac{(2p+1)\pi t}{2\ell}\,dt + \frac{1}{2}\int_{\ell/2}^{3\ell/2} f(t)\cos\frac{(2p+1)\pi t}{2\ell}\,dt\right\} \\ &= \frac{1}{(2p+1)\pi}\left\{\sin\left((2p+1)\pi\cdot\frac{1}{4}\right)+\sin\left((2p+1)\pi\cdot\frac{3}{4}\right)\right\} \\ &= \frac{1}{n\pi}\left\{\sin\left(\frac{n\pi}{4}\right)+\sin\left(n\left(\pi-\frac{\pi}{4}\right)\right)\right\}, \end{aligned}$$

where we have put $2p+1 = n$. Since n is odd, we get

$$\sin\left(n\pi - n\,\frac{\pi}{4}\right) = \cos n\pi\cdot\sin\left(-n\,\frac{\pi}{4}\right) = +\sin\left(n\,\frac{\pi}{4}\right).$$

Then by insertion,

$$a_n = \frac{2}{n\pi}\,\sin\left(n\,\frac{\pi}{4}\right) \qquad \text{for } n \text{ odd.}$$

3) Since $\left|\sin\left(n\,\frac{\pi}{4}\right)\right| = \frac{1}{\sqrt{2}}$, and $\sin\left(n\,\frac{\pi}{4}\right)$ for n odd has changing "double"-sign (two pluses follows by two minuses and vice versa), we get all things considered that

$$f^*(t) = \frac{1}{2}+\frac{\sqrt{2}}{\pi}\left\{\cos\frac{\pi t}{2\ell}+\frac{1}{3}\cos\frac{3\pi t}{2\ell}-\frac{1}{5}\cos\frac{5\pi t}{2\ell}-\frac{1}{7}\cos\frac{7\pi t}{2\ell}++--\cdots\right\}.$$

When $t=0$ we get in particular,

$$f^*(0) = 1 = \frac{1}{2}+\frac{\sqrt{2}}{\pi}\left\{1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}--\cdots\right\},$$

hence by a rearrangement,

$$1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}++--\cdots = \frac{\pi}{\sqrt{2}}\left(1-\frac{1}{2}\right) = \frac{\pi}{2\sqrt{2}}.$$

We get for $t = \frac{\ell}{2}$ the adjusted value $f^*\left(\frac{\ell}{2}\right) = \frac{3}{4}$, thus

$$\begin{aligned} \frac{3}{4} = f^*\left(\frac{\ell}{2}\right) &= \frac{1}{2}+\frac{\sqrt{2}}{\pi}\left\{\cos\frac{\pi}{4}+\frac{1}{3}\cos\frac{3\pi}{4}-\frac{1}{5}\cos\frac{5\pi}{4}-\frac{1}{7}\cos\frac{7\pi}{4}+\cdots\right\} \\ &= \frac{1}{2}+\frac{1}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right\}, \end{aligned}$$

and by a rearrangement,

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots = \frac{\pi}{4}.$$

Remark 1.3 The result is in agreement with that the series on the left hand side is the series for

$\text{Arctan}\ 1 = \frac{\pi}{4}$.

Example 1.23 *The periodic function f of period 2π os given in the interval $]-\pi,\pi]$ by*

$$f(t)=\frac{1}{\pi^4}(t^2-\pi^2)^2, \qquad t\in]-\pi,\pi].$$

1) Sketch the graph of f in the interval $[-\pi,\pi]$.

2) Prove that the Fourier series for f is given by

$$\frac{8}{15}+\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}\cos nt, \qquad t\in\mathbb{R}.$$

Hint: It may be used that

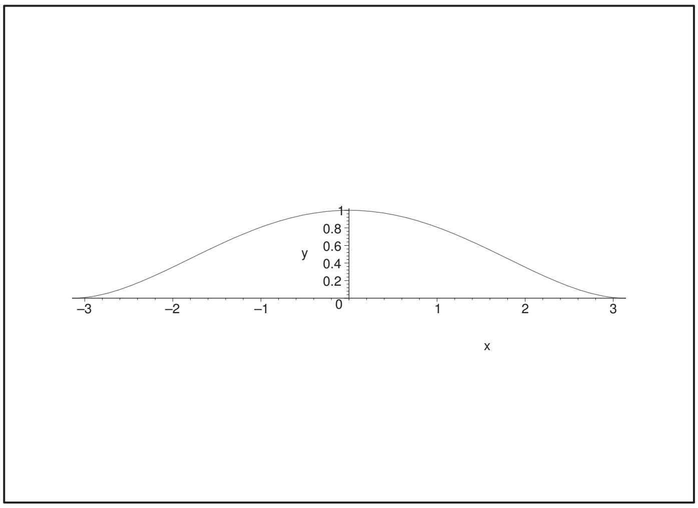
$$\int_0^{\pi}(t^2-\pi^2)^2\cos nt\,dt=24\pi\cdot\frac{(-1)^{n-1}}{n^4}, \qquad n\in\mathbb{N}.$$

3) Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

by using the result of (2).

1) The function $f(t)$ is continuous and piecewise C^1 without vertical half tangents. It follows from $f(-\pi)=f(\pi)=0$ and $f'(t)=\dfrac{4t}{\pi^4}(t^2-\pi^2)$ where $f'(-\pi+)=f'(\pi-)=0$ that we even have that $f(t)$ is everywhere C^1, so $f\in K_{2\pi}^*$. It follows from the **main theorem** that the Fourier series for $f(t)$ is everywhere pointwise convergent and its sum function is $f(t)$.



2) Since $f(t)$ is an even function, the Fourier series is a cosine series. We get for $n = 0$,

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{\pi^4}(t^2-\pi^2)^2\,dt = \frac{2}{\pi^5}\int_0^{\pi}(t^4-2\pi^2t^2+\pi^4)\,dt \\
&= \frac{2}{\pi^5}\left[\frac{1}{5}\,t^5-\frac{2}{3}\,\pi^2t^3+\pi^4t\right]_0^{\pi} = 2\left\{\frac{1}{5}-\frac{2}{3}+1\right\} = \frac{16}{15},
\end{aligned}
$$

hence $\dfrac{1}{2}\,a_0 = \dfrac{8}{15}$.

Furthermore, for $n \in \mathbb{N}$,

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \frac{1}{\pi^4}(t^2-\pi^2)^2\cos nt\,dt = \frac{2}{\pi^5}\int_0^{\pi}(t^2-\pi^2)^2\cos nt\,dt \\
&= \frac{2}{\pi^5}\left[(t^2-\pi^2)^2\cdot\frac{1}{n}\sin nt\right]_0^{\pi} - \frac{2}{\pi^5}\cdot\frac{4}{n}\int_0^{\pi} t(t^2-\pi^2)\sin nt\,dt \\
&= 0+\frac{8}{\pi^5}\cdot\frac{1}{n^2}[t(t^2-\pi^2)\cos nt]_0^{\pi} - \frac{8}{\pi^5}\cdot\frac{1}{n^2}\int_0^{\pi}(3t^2-\pi^2)\cos nt\,dt \\
&= 0-\frac{8}{\pi^5}\cdot\frac{1}{n^3}[(3t^2-\pi^2)\sin nt]_0^{\pi} + \frac{48}{\pi^5}\int_0^{\pi} t\,\sin nt\,dt \\
&= -\frac{48}{\pi^5}\cdot\frac{1}{n^4}[t\cos nt]_0^{\pi} + \frac{48}{\pi^5}\cdot\frac{1}{n^4}\int_0^{\pi}\cos nt\,dt \\
&= \frac{48}{\pi^4}\cdot\frac{1}{n^4}\cdot(-1)^{n-1}+0 = \frac{48}{\pi^4}\cdot\frac{1}{n^4}\cdot(-1)^{n-1}.
\end{aligned}$$

We have proved that the Fourier series is pointwise convergent with an equality sign, cf. (1),

$$(5)\quad f(t) = \frac{8}{15}+\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}\cos nt, \qquad t\in\mathbb{R}.$$

3) In particular, if we choose $t=\pi$ in (5), then

$$0=f(\pi)=\frac{8}{15}+\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}\cos n\pi=\frac{8}{15}-\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Finally, by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{48}\cdot\frac{8}{15}=\frac{\pi^4}{90}.$$

Example 1.24 *1) Sketch the graph of the function* $f(t)=\left|\sin\frac{t}{2}\right|$, $t\in\mathbb{R}$, *in the interval* $[-2\pi,2\pi]$.

2) Prove that

$$f(t)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nt}{4n^2-1},\qquad t\in\mathbb{R}.$$

Hint: One may use without proof that

$$\int\sin\frac{t}{2}\,\cos nt\,dt=\frac{4}{4n^2-1}\left\{n\,\sin\frac{t}{2}\,\sin nt+\frac{1}{2}\cos\frac{t}{2}\,\cos nt\right\},$$

for $t\in\mathbb{R}$ *and* $n\in\mathbb{N}_0$.

3) Find, by using the result of (2), the sum of the series

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{4n^2-1}\qquad og\qquad\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}.$$

1) The function $f(t)$ is continuous and piecewise C^∞ without vertical half tangents. It is also even and periodic with the interval of period $[-\pi,\pi[$. Then by the **main theorem** the Fourier series for $f(t)$ is pointwise convergent everywhere and $f(t)$ is its sum function. Since $f(t)$ is even, the Fourier series is a cosine series.

2) It follows from the above that $b_n=0$ and (cf. the hint)

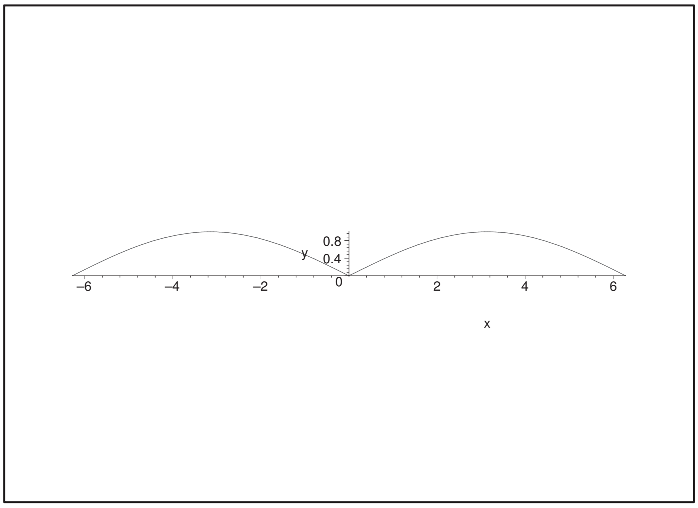
$$\begin{aligned}a_n&=\frac{2}{\pi}\int_0^{\pi}\left|\sin\frac{t}{2}\right|\cos nt\,dt=\frac{2}{\pi}\int_0^{\pi}\sin\frac{t}{2}\cdot\cos nt\,dt\\&=\frac{2}{\pi}\cdot\frac{4}{4n^2-1}\left[n\sin\frac{t}{2}\cdot\sin nt+\frac{1}{2}\cos\frac{t}{2}\,\cos nt\right]_0^{\pi}=\frac{4}{\pi}\cdot\frac{-1}{4n^2-1}.\end{aligned}$$

In particular,

$$\frac{1}{2}\,a_0=\frac{1}{\pi}\int_0^{\pi}\sin\frac{t}{2}\,dt=\frac{1}{\pi}\left[-\cos\frac{t}{2}\right]_0^{\pi}=\frac{2}{\pi},$$

so we get the Fourier expansion with pointwise equality sign, cf. (1),

$$f(t)=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nt}{4n^2-1},\qquad t\in\mathbb{R}.$$



3) a) If we insert $t = 0$ into the Fourier series, we get

$$f(0) = 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{4}{\pi}\left\{\frac{1}{2} - \sum_{n=1}^{\infty}\frac{1}{4n^2-1}\right\},$$

hence by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

Alternatively, it follows by a decomposition that

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\cdot\frac{1}{2n-1} - \frac{1}{2}\cdot\frac{1}{2n+1}.$$

The corresponding segmental sequence is then

$$\begin{aligned} s_N &= \sum_{n=1}^{N}\frac{1}{4n^2-1} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{2n+1} \\ &= \frac{1}{2}\sum_{n=1}^{N}\frac{1}{2n-1} - \frac{1}{2}\sum_{n=2}^{N+1}\frac{1}{2n-1} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2N+1} \\ &\to \frac{1}{2} \qquad \text{for } N \to \infty, \end{aligned}$$

and the series is convergent with the sum

$$\sum_{n=1}^{\infty} = \lim_{N\to\infty} s_N = \frac{1}{2}.$$

b) When we insert $t = \pi$ into the Fourier series, we get

$$f(\pi) = 1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{4}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1}\right\}.$$

Hence by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}.$$

Alternatively, we get (cf. the decomposition above) the segmental sequence,

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{(-1)^{n-1}}{4n^2-1} = \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^{n-1}}{2n-1} - \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^{n-1}}{2n+1} \\
&= \frac{1}{2}\sum_{n=0}^{N-1} \frac{(-1)^{n}}{2n+1} + \frac{1}{2}\sum_{n=1}^{N} \frac{(-1)^{n}}{2n+1} \\
&= \sum_{n=0}^{N-1} \frac{(-1)^{n}}{2n+1}\cdot 1^{2n+1} - \frac{1}{2} + \frac{1}{2}\cdot\frac{(-1)^N}{2N+1} \\
&\to \quad \text{Arctan}\, 1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2} \qquad \text{for } N \to \infty.
\end{aligned}$$

The series is therefore convergent with the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1} = \lim_{N\to\infty}\sum_{n=1}^{N} \frac{(-1)^{n-1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}.$$

Example 1.25 *Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by*

$$f(x) = \frac{1}{5-3\cos x}, \qquad x \in \mathbb{R}.$$

Prove that $f(x)$ has the Fourier series

$$\frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{3^n}\cos nx, \qquad x \in \mathbb{R}.$$

Let the function $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = \frac{\sin x}{5-3\cos x}, \qquad x \in \mathbb{R}.$$

Prove that $g(x)$ has the Fourier series

$$\frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{3^n}\sin nx, \qquad x \in \mathbb{R}.$$

1) Explain why the Fourier series for f can be differentiated termwise, and find the sum of the differentiated series for $x = \frac{\pi}{2}$.

2) Find by means of the power series for $\ln(1-x)$ the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\cdot 3^n}.$$

3) Prove that the Fourier series for g can be integrated termwise in $\mathbb{R}$.

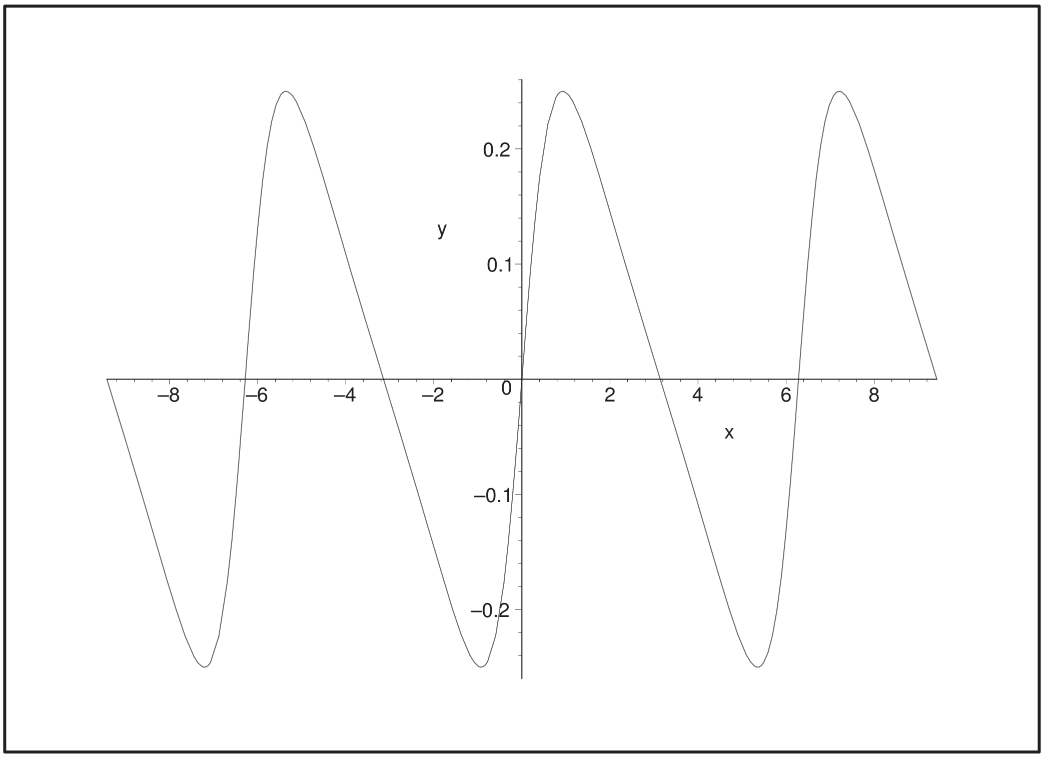
4) Finally, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\cdot 3^n}\cos nx, \qquad x \in \mathbb{R}.$$

Since $\left|\dfrac{e^{ix}}{3}\right| = \dfrac{1}{3} < 1$ for every $x \in \mathbb{R}$, we get by the complex quotient series (the proof for the legality of this procedure is identical with the proof in the real case),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{3^n}\, e^{inx} &= \sum_{n=1}^{\infty} \left(\frac{e^{ix}}{3}\right)^n = \frac{e^{ix}}{3} \cdot \frac{1}{1 - \dfrac{e^{ix}}{3}} = \frac{e^{ix}}{3 - e^{ix}} \cdot \frac{3 - e^{-ix}}{3 - e^{-ix}} \\ &= \frac{3e^{ix} - 1}{9 - 6\cos x + 1} = \frac{1}{2} \cdot \frac{3\cos x - 1 + 3i\sin x}{5 - 3\cos x}.\end{aligned}$$

Hence



$$\begin{aligned}\frac{1}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{3^n} \cos nx &= \frac{1}{4} + \frac{1}{2} \operatorname{Re}\left\{ \sum_{n=1}^{\infty} \frac{1}{3^n}\, e^{inx} \right\} = \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3\cos x - 1}{5 - 3\cos x} \\ &= \frac{1}{4} \cdot \frac{(5 - 3\cos x) + (3\cos x - 1)}{5 - 3\cos x} = \frac{1}{5 - 3\cos x} = f(x),\end{aligned}$$

and

$$\frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{3^n} \sin nx = \frac{2}{3} \operatorname{Im}\left\{ \sum_{n=1}^{\infty} \frac{1}{3^n}\, e^{inx} \right\} = \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{3\sin x}{5 - 3\cos x} = \frac{\sin x}{5 - 3\cos x} = g(x).$$

1) From

$$\left|\frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{3^n}\,\cos nx\right| \le \frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{4}+\frac{1}{4}=\frac{1}{2},$$

follows that the Fourier series has a convergent majoring series, so it is uniformly convergent with the continuous sum function

$$\frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{3^n}\,\cos nx = \frac{1}{5-3\cos x} = f(x).$$

The termwise differentiated series,

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{3^n}\,\sin nx,$$

is also uniformly convergent, because $\sum n/3^n < \infty$ is a convergent majoring series. Then it follows from the theorem of differentiation of series that the differentiated series is convergent with the sum function

$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{3^n}\,\sin nx = f'(x) = \frac{d}{dx}\left(\frac{1}{5-3\cos x}\right) = -\frac{3\sin x}{(5-3\cos x)^2}.$$

Hence for $x=\frac{\pi}{2}$,

$$\begin{aligned}-\frac{3}{25} &= -\frac{1}{2}\sum_{n=1}^{\infty}\frac{n}{3^n}\,\sin\frac{n\pi}{2} = -\frac{1}{2}\sum_{m=0}^{\infty}\frac{4m+1}{3^{4m+1}}+\frac{1}{2}\sum_{m=0}^{\infty}\frac{4m+3}{3^{4m+3}}\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2n+1}{3^{2n+1}} = \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2n+1}{9^n}.\end{aligned}$$

2) It follows from

$$\ln\left(\frac{1}{1-x}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}, \qquad |x|<1,$$

that we for $x=\frac{1}{3}$ have

$$\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n} = \ln\frac{3}{2}.$$

3) Since also $\frac{2}{3}\sum 3^{-n}\sin nx$ is uniformly convergent (same argument as in (1), i.e. the obvious majoring series is convergent), it follows that the Fourier series for g can be integrated termwise in $\mathbb{R}$.

4) We get by termwise integration that

$$\begin{aligned}\int_0^x g(t)\,dt &= \int_0^x \frac{\sin t}{5-3\cos t}\,dt = \frac{1}{3}[\ln(5-3\cos t)]_0^x = \frac{1}{3}\ln(5-3\cos x) - \frac{1}{3}\ln 2\\ &= \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{3^n}\int_0^x \sin nx\,dx = -\frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}(\cos nx - 1),\end{aligned}$$

hence by a rearrangement,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n}\cos nx &= \sum_{n=1}^{\infty}\frac{1}{n\cdot 3^n} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln(5-3\cos x) = \ln\frac{3}{2} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln(5-3\cos x)\\ &= \ln 3 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln(5-3\cos x).\end{aligned}$$

Example 1.26 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = \begin{cases} t, & \text{for } 0 < t \leq \pi, \\ \\ 0, & \text{for } \pi < t \leq 2\pi. \end{cases}$$

1) Sketch the graph of f in the interval $[-2\pi, 2\pi]$.

2) Prove that the Fourier series for f is given by

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{\pi n^2} \cos nt + \frac{(-1)^{n-1}}{n} \sin nt \right), \qquad t \in \mathbb{R}.$$

Hint: One may without proof apply that for every $n \in \mathbb{N}$,

$$\int t \cos nt \, dt = \frac{1}{n^2}(nt \sin nt + \cos nt),$$

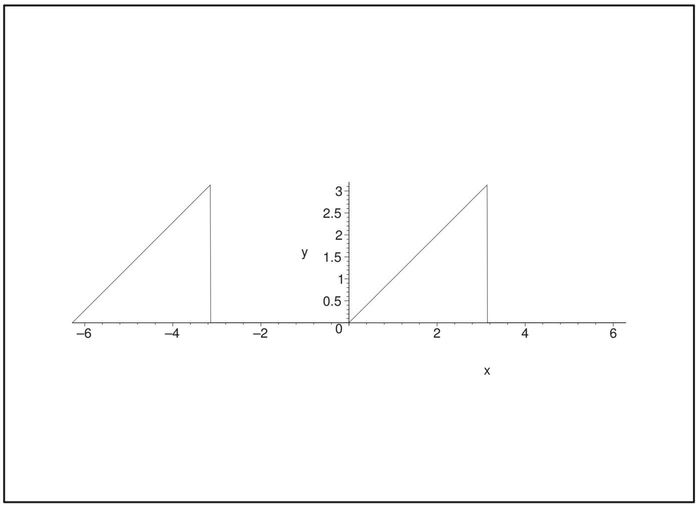
$$\int t \sin nt \, dt = \frac{1}{n^2}(-nt \cos nt + \sin nt).$$

3) Find the sum of the Fourier series for $t = \pi$.

1) The adjusted function is

$$f^*(t) = \begin{cases} t, & \text{for } 0 < t < \pi, \\ \pi/2, & \text{for } t = \pi, \\ 0, & \text{for } \pi < t \leq 2\pi, \end{cases}$$

continued periodically.



Since $f(t)$ is piecewise C^1,

$$f'(t) = \begin{cases} 1 & \text{for } 0 < t < \pi, \\ 0 & \text{for } \pi < t < 2\pi, \end{cases}$$

without vertical half tangents, it follows from the **main theorem** that the Fourier series is pointwise convergent with the adjusted function $f^*(t)$ as its sum function. In particular, $f \sim$ can be replaced by $f^*(t) = .$

2) The Fourier series is pointwise

$$f^*(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n \cos nt + b_n \sin nt\},$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt = \frac{1}{\pi n^2}[nt\sin nt + \cos nt]_0^{\pi} \\ &= \frac{1}{\pi n^2}\{0+\cos n\pi - 1\} = \frac{(-1)^n-1}{\pi n^2} \qquad \text{for } n \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt = \frac{1}{\pi n^2}[-nt\cos nt + \sin nt]_0^{\pi} \\ &= \frac{1}{\pi n^2}\{-n\pi\cos n\pi + 0 + 0 - 0\} = \frac{1}{n}\cdot(-1)^{n-1} \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

Finally, we consider the exceptional value $n = 0$, where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\infty} = \frac{\pi}{2}.$$

Hence by insertions,

$$f^*(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n-1}{\pi n^2}\,\cos nt + \frac{(-1)^{n-1}}{n}\,\sin nt\right\}, \quad t \in \mathbb{R}.$$

3) The argument is given in (1), so the sum is

$$f^*(\pi) = \frac{\pi}{2}.$$

Alternatively,

$$\begin{aligned} \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n-1}{\pi n^2}\,\cos n\pi + \frac{(-1)^{n-1}}{n}\,\sin n\pi\right\} &= \frac{\pi}{4} + \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{-2}{(2p+1)^2}\,\cos(2p+1)\pi \\ = \frac{\pi}{4} + \frac{2}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}. \end{aligned}$$

Notice that every $n \in \mathbb{N}$ is uniquely written in the form $n = (2p+1)\cdot 2^q$, thus

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\sum_{q=0}^{\infty}\frac{1}{(2^2)^q} = \frac{1}{1-\frac{1}{4}}\cdot\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2} = \frac{4}{3}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2},$$

and hence

$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

We get by insertion the sum

$$\frac{\pi}{4} + \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi}{4} + \frac{2}{\pi} \cdot \frac{\pi^2}{8} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

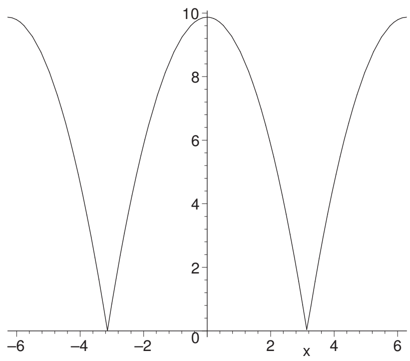
2 Fourier series and uniform convergence

Example 2.1 *The function $f \in K_{2\pi}$ is given by*

$$f(t) = \pi^2 - t^2, \qquad -\pi < t \le \pi.$$

1) Find the Fourier series for f.

2) Find the sum function of the Fourier series and prove that the Fourier series is uniformly convergent in $\mathbb{R}$.



No matter the formulation of the problem, it is *always* a good idea to start by sketching the graph of the function over at periodic interval and slightly into the two neighbouring intervals.

Then *check* the assumptions of the **main theorem**: Clearly, $f \in C^1(]-\pi,\pi[)$ without vertical half tangents, hence $f \in K_{2\pi}^*$.

The Fourier series is *pointwise convergent* everywhere, so $\sim$ can be replaced by $=$ when we use the *adjusted* function

$$f^*(t) = \frac{f(t+) + f(t-)}{2}$$

as our sum function.

It follows from the graph that $f(t)$ is continuous everywhere, hence $f^*(t) = f(t)$, and we have obtained without any calculation that we have pointwise everywhere

$$f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}.$$

After this simple introduction with lots of useful information we start on the task itself.

1) The function f is *even*, $(f(-t) = f(t))$, so $b_n = 0$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - t^2)\cos nt\,dt.$$

We must not divide by 0, so let $n \neq 0$. Then we get by a couple of partial integrations,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi}(\pi^2-t^2)\cos nt\,dt = \frac{2}{\pi n}\left[(\pi^2-t^2)\sin nt\right]_0^{\pi} + \frac{4}{\pi n}\int_0^{\pi} t\,\sin nt\,dt \\ &= 0+\frac{4}{\pi n^2}[-t\,\cos nt]_0^{\pi} + \frac{4}{\pi n^2}\int_0^{\pi}\cos nt\,dt = -\frac{4}{\pi n^2}\,\pi\,\cos n\pi + 0 = \frac{4(-1)^{n+1}}{n^2}. \end{aligned}$$

In the exceptional case $n = 0$ we get instead

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi^2-t^2)dt = \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{4\pi^3}{3\pi} = \frac{4\pi^2}{3}.$$

The Fourier series is then, where we already have argued for the equality sign,

$$(6)\quad f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nt = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{4}{n^2}\,\cos nt.$$

2) The estimate $\left|(-1)^{n-1}\,\frac{4}{n^2}\,\cos nt\right| \leq \frac{4}{n^2}$ shows that

$$\frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{3} + 4\cdot\frac{\pi^2}{6} = \frac{4\pi^2}{3}$$

is a convergent majoring series. Hence the Fourier series is *uniformly convergent.*

Remark 2.1 We note that if we put $t = 0$ into (6), then

$$f(0) = \pi^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2},$$

and hence by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Example 2.2 *A function $f \in K_{2\pi}$ is given in the interval $]0, 2\pi]$ by*

$$f(t) = t^2.$$

Notice the given interval!

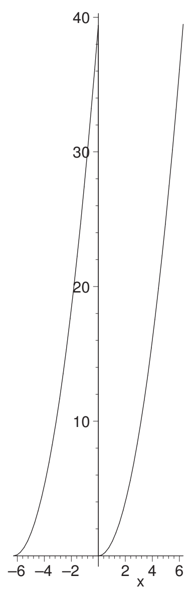
1) *Sketch the graph of f in the interval $]\,2\pi, 2\pi]$.*

2) *Sketch the graph of the sum function of the Fourier series in the interval $]-2\pi, 2\pi]$, and check if the Fourier series is uniformly convergent in $\mathbb{R}$.*

3) *Explain why we have for every function $F \in K_{2\pi}$,*

$$\int_{-\pi}^{\pi} F(t)\,dt = \int_{0}^{2\pi} F(t)\,dt.$$

The find the Fourier series for f.



1) The graph is sketch on the figure. It is not easy to sketch the adjusted function $f^*(1)$ in MAPLE, so we shall only give the definition,

$$f^*(t) = \begin{cases} f(t) & \text{for } t - 2n\pi \in]0, 2\pi[, \quad n \in \mathbb{Z}, \\[2mm] 2\pi^2 & \text{for } t = 2n\pi, \quad n \in \mathbb{Z}. \end{cases}$$

2) Since f is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is *pointwise convergent* everywhere, and its sum function is the adjusted function $f^*(t)$.

   Each term of the Fourier series is continuous, while the sum function $f^*(t)$ is *not* continuous. Hence, it follows that the Fourier series *cannot* be uniformly convergent in $\mathbb{R}$.

3) When $F \in K_{2\pi}$, then F is periodic of period 2π, hence

$$\int_{-\pi}^{\pi} F(t)\,dt = \int_0^{\pi} F(t)\,dt + \int_{-\pi}^{0} F(t+2\pi)\,dt = \int_0^{\pi} F(t)\,dt + \int_{\pi}^{2\pi} F(t)\,dt = \int_0^{2\pi} F(t)\,dt.$$

   In particular,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt, \quad \text{og} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt.$$

   Thus

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} t^2\,dt = \frac{8\pi^3}{3\pi} = \frac{8\pi^2}{3},$$

   and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt\,dt = \frac{1}{\pi n}\left[t^2 \sin t\right]_0^{2\pi} - \frac{2}{\pi n}\int_0^{2\pi} t \sin nt\,dt \\ &= 0 + \frac{2}{\pi n^2}[t\cos nt]_0^{2\pi} - \frac{2}{\pi n^2}\int_0^{2\pi} \cos nt\,dt = \frac{2}{\pi n^2}\cdot 2\pi = \frac{4}{n^2} \end{aligned}$$

   for $n \geq 1$, and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt\,dt = \frac{1}{\pi n}\left[-t^2 \cos nt\right]_0^{2\pi} + \frac{2}{\pi n}\int_0^{2\pi} t \cos nt\,dt \\ &= -\frac{4\pi^2}{\pi n} + \frac{2}{\pi n^2}[t\sin nt]_0^{2\pi} - \frac{2}{\pi n^2}\int_0^{2\pi} \sin nt\,dt = -\frac{4\pi}{n} - \frac{2}{\pi n^3}[\cos nt]_0^{2\pi} = -\frac{4\pi}{n}. \end{aligned}$$

   The Fourier series is (NB. Remember the term $\frac{1}{2}\,a_0$)

$$f \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2}\,\cos nt - \frac{4\pi}{n}\,\sin nt\right\}$$

   (convergence in "energy") and

$$f^*(t) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2}\,\cos nt - \frac{4\pi}{n}\,\sin nt\right\}$$

   (pointwise convergence).

Example 2.3 *Let the function* $f \in K_{2\pi}$ *be given by*

$$f(t) = e^t \sin t \qquad \text{for } -\pi < t \le \pi.$$

1) Prove that the Fourier series for f *is given by*

$$\frac{\sinh \pi}{\pi}\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{(-1)^n(4-2n^2)}{n^4+4}\cos nt+\frac{(-1)^{n-1}4n}{n^4+4}\sin nt\right)\right\}.$$

We may use the following formulæ without proof:

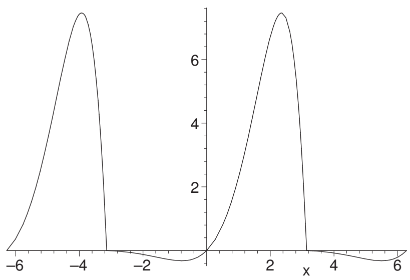
$$\int_{-\pi}^{\pi} e^t \cos mt\, dt = \frac{2(-1)^m \sinh \pi}{1+m^2}, \qquad m \in \mathbb{N}_0,$$

$$\int_{\pi}^{\pi} e^t \sin mt\, dt = \frac{2m(-1)^{m+1} \sinh \pi}{1+m^2}, \qquad m \in \mathbb{N}_0.$$

2) Prove that the Fourier series in 1. is uniformly convergent.

3) Find by means of the result of 1. the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^2-2}{n^4+4}.$$



The function $f(t)$ is continuous and piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is pointwise convergent everywhere, and its sum function is $f^*(t) = f(t)$.

1) By using *complex* calculations, where

$$\sin t = \frac{1}{2i}\left(e^{it}-e^{-it}\right),$$

and $b_0 = 0$, we get that

$$\begin{aligned}
a_n + ib_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^t \sin t \cdot e^{int}\,dt = \frac{1}{2i\pi}\int_{-\pi}^{\pi}\left\{e^{(1+i(n+1))t} - e^{(1+i(n-1))t}\right\}dt \\
&= \frac{1}{2\pi i}\left[\frac{1}{1+i(n+1)}\{(-1)^{n+1}(e^{\pi}-e^{-\pi})\} - \frac{1}{1+i(n-1)}\{(-1)^{n-1}(e^{\pi}-e^{-\pi})\}\right] \\
&= \frac{\sinh\pi}{\pi}\cdot i(-1)^n\left\{\frac{1}{1+i(n+1)} - \frac{1}{1+i(n-1)}\right\} \\
&= \frac{\sinh\pi}{\pi}\cdot(-1)^n\, i\cdot\frac{1+i(n-1)-\{1+i(n+1)\}}{1-(n^2-1)+i\,2n} \\
&= \frac{\sinh\pi}{\pi}\,(-1)^n\cdot i\cdot\frac{-2i}{-(n^2-2)+2in} = \frac{\sinh\pi}{\pi}\cdot(-1)^n\cdot 2\cdot\frac{-(n^2-2)-2in}{(n^2-2)^2+4n^2} \\
&= \frac{\sinh\pi}{\pi}\cdot(-1)^n\cdot\frac{4-2n^2-4in}{n^4+4}.
\end{aligned}$$

When we split into the real and the imaginary part we find

$$a_n = \frac{\sinh \pi}{\pi} \cdot (-1)^n \cdot \frac{4-2n^2}{n^4+4}, \qquad n \geq 0,$$

and

$$b_n = \frac{\sinh \pi}{\pi} \cdot (-1)^n \cdot \frac{4n}{n^4+4}, \qquad n \geq 1.$$

Alternatively we get by *real* computations,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^t \sin t\, \cos nt\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t\{\sin(n+1)t - \sin(n-1)t\}\, dt, \\
&= \frac{1}{2\pi}\left\{\frac{2(n+1)\cdot(-1)^{n+2}\sinh \pi}{1+(n+1)^2} - \frac{2(n-1)(-1)^n \sinh \pi}{1+(n-1)^2}\right\} \\
&= \frac{\sinh \pi}{\pi}\,(-1)^n\,\frac{(n+1)(n^2-2n+2)-(n-1)(n^2+2n+2)}{(n^2+2-2n)(n^2+2+2n)} \\
&= \frac{\sinh \pi}{\pi} \cdot (-1)^n \cdot \frac{4-2n^2}{n^4+4},
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^t \sin t\, \sin nt\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^t\{\cos(n-1)t - \cos(n+1)t\}\, dt \\
&= \frac{1}{2\pi}\left\{\frac{2(-1)^{n-1}\sinh \pi}{1+(n-1)^2} - \frac{2(-1)^{n-1}\sinh \pi}{1+(n+1)^2}\right\} = \frac{\sinh \pi}{\pi} \cdot (-1)^{n-1} \cdot \frac{n^2+2n+2-(n^2-2n+2)}{(n^2-2n+2)(n^2+2n+2)} \\
&= \frac{\sinh \pi}{\pi} \cdot (-1)^{n-1} \cdot \frac{4n}{n^4+4}.
\end{aligned}$$

In both cases we see that $a_0 = \dfrac{\sinh \pi}{\pi}$, hence we get the Fourier series (with equality by the remarks above)

$$f(t) = \frac{\sinh \pi}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n(4-2n^2)}{n^4+4}\cos nt + \frac{(-1)^{n-1}4n}{n^4+4}\sin nt\right)\right\}.$$

2) The Fourier series has the convergent majoring series

$$\frac{\sinh \pi}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{2n^2+4}{n^4+4} + \sum_{n=1}^{\infty}\frac{4n}{n^4+4}\right\}$$

(the difference of the degrees of the denominator and the numerator is ≥ 2), hence the Fourier series is *uniformly convergent*.

3) By choosing $t = \pi$ we get the pointwise result,

$$f(\pi) = 0 = \frac{\sinh \pi}{\pi}\left\{\frac{1}{2} - 2\sum_{n=1}^{\infty}\frac{n^2-2}{n^4+4}\right\},$$

hence by a rearrangement

$$\sum_{n=1}^{\infty} \frac{n^2-2}{n^4+4} = \frac{1}{4}.$$

Alternatively it follows by a decomposition,

$$\frac{n^2-2}{n^4} = \frac{n^2-2}{(n^2-2n+2)(n^2+2n+2)} = \frac{1}{2}\,\frac{n-1}{1+(n-1)^2} - \frac{1}{2}\,\frac{n+1}{1+(n+1)^2},$$

so the sequential sequence is a telescoping sequence,

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} \frac{n^2-2}{n^4+4} = \frac{1}{2}\sum_{n=1}^{N} \frac{n-1}{1+(n-1)^2} - \sum_{n=1}^{N} \frac{n+1}{1+(n+1)^2} \\
&= \frac{1}{2}\sum_{\substack{n=0\\(n=1)}}^{N-1} \frac{n}{1+n^2} - \frac{1}{2}\sum_{n=2}^{N+1} \frac{n}{1+n^2} \\
&= \frac{1}{2}\cdot\frac{1}{1+1^2} - \frac{1}{2}\cdot\frac{N}{1+N^2} - \frac{1}{2}\cdot\frac{N+1}{1+(N+1)^2} \to \frac{1}{4} \qquad \text{for } N \to \infty,
\end{aligned}$$

and it follows by the definition that

$$\sum_{n=1}^{\infty} \frac{n^2-2}{n^4+4} = \lim_{N\to\infty} s_N = \frac{1}{4}.$$

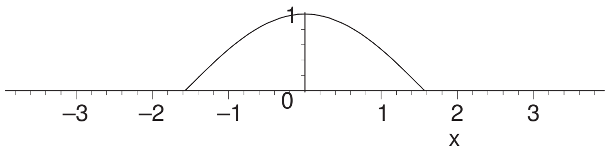
Example 2.4 *Find the Fourier series for the function $f \in K_{2\pi}$, which is given in the interval $]-\pi,\pi]$ by*

$$f(t) = \begin{cases} 0, & \textit{for } -\pi < t < -\pi/2, \\[2ex] \cos t, & \textit{for } -\pi/2 \le t \le \pi/2, \\[2ex] 0, & \textit{for } \pi/2 < t \le \pi. \end{cases}$$

Prove that the series is absolutely and uniformly convergent in the interval $\mathbb{R}$ and find for $t \in [-\pi,\pi]$ the sum of the termwise integrated series from 0 to t. Then find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)(4n+2)(4n+3)}$$

The function f is continuous and piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$. The Fourier series is by the **main theorem** pointwise convergent with the sum function $f^*(t) = f(t)$.



Since $f(t)$ is *even*, all $b_n = 0$. For $n \neq 1$ we get

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi/2} \cos t \cdot \cos nt\, dt = \frac{1}{\pi}\int_0^{\pi/2}\{\cos(n+1)t+\cos(n-1)t\}dt \\
&= \frac{1}{\pi}\left\{\frac{1}{n+1}\,\sin\left((n+1)\frac{\pi}{2}\right)+\frac{1}{n-1}\,\sin\left((n-1)\frac{\pi}{2}\right)\right\} \\
&= \frac{1}{\pi}\left\{\frac{1}{n+1}\,\cos\left(\frac{n\pi}{2}\right)-\frac{1}{n-1}\,\cos\left(\frac{n\pi}{2}\right)\right\} = -\frac{2}{\pi}\,\frac{1}{n^2-1}\,\cos\left(\frac{n\pi}{2}\right).
\end{aligned}$$

It follows that $a_{2n+1} = 0$ for $n \in \mathbb{N}$ and that

$$a_{2n} = -\frac{2}{\pi}\,\frac{(-1)^n}{4n^2-1}, \qquad \text{for } n \in \mathbb{N}_0, \quad \text{in particular } a_0 = \frac{2}{\pi} \text{ for } n = 0.$$

In the exceptional case $n = 1$ we get instead

$$a_1 = \frac{2}{\pi}\int_0^{\pi/2} \cos^2 t\, dt = \frac{1}{\pi}\int_0^{\pi/2}\{\cos 2t + 1\}\, dt = \frac{1}{2}.$$

The Fourier series becomes with an equality sign according to the above,

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\,\cos t + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1}\,\cos 2nt.$$

The Fourier series has the convergent majoring series

$$\frac{1}{\pi} + \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1},$$

so it is absolutely and uniformly convergent. Therefore, we can integrate it termwise, and we get

$$\int_0^t f(\tau)\, d\tau = \frac{t}{\pi} + \frac{1}{2}\,\sin t + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)2n(2n+1)}\,\sin 2nt,$$

which also is equal to

$$\int_0^t f(\tau)\, d\tau = \begin{cases} -1, & \text{for } -\pi < t < -\dfrac{\pi}{2}, \\ \sin t, & \text{for } -\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}, \\ 1, & \text{for } \dfrac{\pi}{2} < t < \pi. \end{cases}$$

By choosing $t = \dfrac{\pi}{4}$ we get

$$\begin{aligned}
\int_0^{\pi/4} f(\tau)\,d\tau = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} &= \frac{1}{4}+\frac{\sqrt{2}}{4}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)2n(2n+1)}\,\sin\left(\frac{n\pi}{2}\right)\\
&= \frac{1}{4}+\frac{\sqrt{2}}{4}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(2n+1)(2n+2)(2n+3)}\,\sin\left(n\,\frac{\pi}{2}+\frac{\pi}{2}\right)\\
&= \frac{1}{4}+\frac{\sqrt{2}}{4}+\frac{2}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^{2p}}{4p+1)(4p+2)(4p+3)}\,\sin\left(p\pi+\frac{\pi}{2}\right)\\
&= \frac{1}{4}+\frac{\sqrt{2}}{4}+\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(4n+1)(4n+2)(4n+3)},
\end{aligned}$$

where we have a) changed index, $n \mapsto n+1$, and b) noticed that we only get contributions for $n = 2p$ *even*.

Finally, we get by a rearrangement,

$$\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(4n+1)(4n+2)(4n+3)} = \frac{\pi}{2}\left(\frac{\sqrt{2}}{2}-\frac{1}{4}-\frac{\sqrt{2}}{4}\right) = \frac{\pi(\sqrt{2}-1)}{8}.$$

Example 2.5 *Find the Fourier series for the function $f \in K_{2\pi}$, which is given in the interval $]-\pi,\pi[$ by*

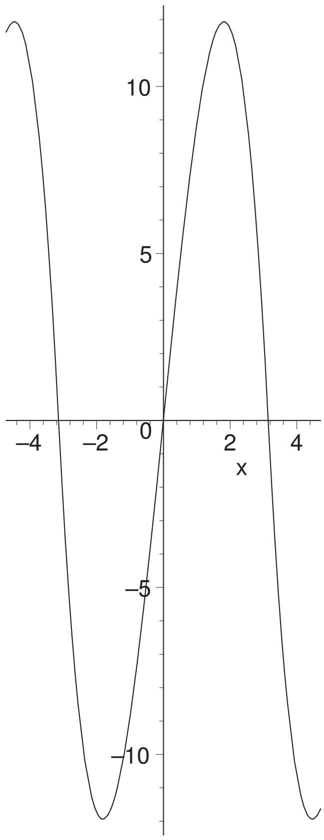
$$f(t) = t(\pi^2 - t^2).$$

Prove that the Fourier series is uniformly convergent in the interval $\mathbb{R}$, and find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}.$$

The function $f \in C^\infty(]-\pi,\pi[)$ is without vertical half tangents, so $f \in K_{2\pi}^*$. Furthermore, f is *odd*, so the Fourier series is a sine series, thus $a_n = 0$. The periodic continuation is continuous, so the adjusted function $f^*(t) = f(t)$ is by the **main theorem** the pointwise sum function for the Fourier series, and we can replace $\sim$ by an equality sign,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt, \qquad t \in \mathbb{R}.$$



We obtain by some partial integrations,

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi (\pi^2-t^3)\sin nt\,dt = -\frac{2}{\pi n}\left[t(\pi^2-t^2)\cos nt\right]_0^\pi + \frac{2}{\pi n}\int_0^\pi (\pi^2-3t^2)\cos nt\,dt \\
&= \frac{2}{\pi n^2}\left[\left(\pi^2-3t^2\right)\sin nt\right]_0^\pi + \frac{12}{\pi n^2}\int_0^\pi t\,\sin nt\,dt = \frac{12}{\pi n^3}[-t\,\cos nt]_0^\pi + \frac{12}{\pi n^3}\int_0^\pi \cos nt\,dt \\
&= \frac{12\pi}{\pi n^3}\cdot(-1)^{n+1} = \frac{12}{n^3}\cdot(-1)^{n+1}.
\end{aligned}$$

The Fourier series is then

$$f(t) = 12\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^3}\,\sin nt.$$

The Fourier series has the convergent majoring series

$$12\sum_{n=1}^\infty \frac{1}{n^3},$$

so it is uniformly convergent in $\mathbb{R}$.

If we put $t = \dfrac{\pi}{2}$, we get

$$\begin{aligned}
f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2}\left(\pi^2-\frac{\pi^2}{4}\right) = \frac{3\pi^2}{8} = 12\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^3}\,\sin n\,\frac{\pi}{n} \\
&= 12\sum_{p=1}^\infty \frac{(-1)^{2p}}{(2p-1)^3}\,\sin\left(p\pi-\frac{\pi}{2}\right) = 12\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n-1)^3}.
\end{aligned}$$

Then by a rearrangement,

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

Example 2.6 *Let $f \in K_{2\pi}$ be given in the interval $[-\pi, \pi]$ by*

$$f(t) = \begin{cases} \sin 2t, & \text{for } |t| \le \dfrac{\pi}{2}, \\ \\ 0, & \text{for } \dfrac{\pi}{2} < |t| \le \pi. \end{cases}$$

1) Prove that f has the Fourier series

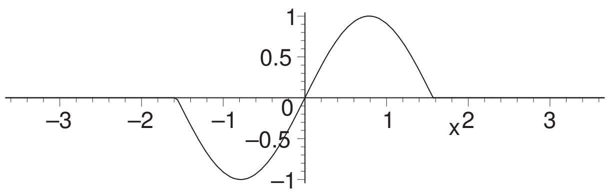
$$\frac{1}{2}\sin 2t + \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+3)}\sin(2n+1)t,$$

and prove that it is uniformly convergent in the interval $\mathbb{R}$.

2) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)(2n+3)}.$$

The function f is continuous and piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$. The Fourier series is then by the **main theorem** pointwise convergent with sum $f^*(t) = f(t)$.



1) Since f is *odd*, we have $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} \sin 2t\,\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi/2}\{\cos(n-2)t - \cos(n+2)t\}dt.$$

For $n = 2$ we get in particular,

$$b_2 = \frac{1}{\pi}\int_0^{\pi/2}(1-\cos 4t)dt = \frac{1}{\pi}\cdot\frac{\pi}{2} - 0 = \frac{1}{2}.$$

For $n \in \mathbb{N}\setminus\{2\}$ we get

$$b_n = \frac{1}{\pi}\left[\frac{\sin(n-2)t}{n-2} - \frac{\sin(n+2)t}{n+2}\right]_0^{\pi/2} = \frac{1}{\pi}\left\{\frac{\sin\left(\dfrac{n}{2}-1\right)\pi}{n-2} - \frac{\sin\left(\dfrac{n}{2}+1\right)\pi}{n+2}\right\}.$$

In particular, $b_{2n} = 0$ for $n \ge 2$, and

$$\begin{aligned} b_{2n+1} &= \frac{1}{\pi}\left\{\frac{\sin\left(n-\dfrac{1}{2}\right)\pi}{2n-1} - \frac{\sin\left(n+2-\dfrac{1}{2}\right)\pi}{2n+3}\right\} = \frac{(-1)^{n+1}}{\pi}\left\{\frac{1}{2n-1} - \frac{1}{2n+3}\right\} \\ &= \frac{4}{\pi}\cdot(-1)^{n+1}\cdot\frac{1}{(2n-1)(2n+3)} \qquad \text{for } n \ge 0. \end{aligned}$$

The Fourier series is (with equality, cf. the above)

$$f(t) = \frac{1}{2}\sin 2t + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n-1)(2n+3)}\sin(2n+1)t.$$

The Fourier series has the convergent majoring series

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n-1)(2n+3)},$$

so it is uniformly convergent.

2) By a comparison we see that we are missing a factor $2n+1$ in the denominator. We can obtain this by a termwise integration of the Fourier series, which is legal now due to the uniform convergence),

$$\int_0^t f(\tau)\,d\tau = \frac{1}{4} - \frac{1}{4}\cos 2t + \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)t}{(2n-1)(2n+1)(2n+3)} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n-1)(2n+1)(2n+3)}.$$

By choosing $t = \dfrac{\pi}{2}$, the first series is 0, hence

$$\begin{aligned}\int_0^{\pi/2} f(\tau)\,d\tau &= \int_0^{\pi/2} \sin 2\tau\,d\tau = \left[-\frac{1}{2}\,\cos 2\tau\right]_0^{\pi/2} = 1\\ &= \frac{1}{4}+\frac{1}{4}+0-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)(2n+3)},\end{aligned}$$

and by a rearrangement,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(2n+1)(2n+3)} = \frac{\pi}{4}\left(\frac{1}{2}-1\right) = -\frac{\pi}{8}.$$

Alternatively we may apply the following method (only sketched here):

a) We get by a decomposition,

$$\frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{1}{8}\left\{\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)-\left(\frac{1}{2n+3}-\frac{1}{2n+3}\right)\right\}.$$

b) The segmental sequence becomes

$$\begin{aligned}s_N &= \sum_{n=0}^{N}\frac{(-1)^n}{(2n-1)(2n+1)(2n+3)}\\ &= -\frac{1}{4}+\frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^n}{2n-1}+\frac{1}{4}+\frac{1}{4}\cdot\frac{(-1)^{N+1}}{2N+1}+\frac{1}{8}\cdot(-1)^{N+1}\left\{\frac{1}{2N+1}-\frac{1}{2N+3}\right\}.\end{aligned}$$

c) Finally, by taking the limit,

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)(2n+3)} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}-\frac{1}{2}\text{ Arctan }1 = -\frac{\pi}{8}.$$

Example 2.7 *Given the periodic function $f:\mathbb{R}\to\mathbb{R}$ of period 2π, which is given in the interval $[-\pi,\pi]$ by*

$$f(t)=\begin{cases} |\sin 2t|, & 0\le |t|\le \dfrac{\pi}{2},\\[2ex] 0, & \dfrac{\pi}{2}<|t|\le\pi.\end{cases}$$

1) Prove that the Fourier series for f can be written

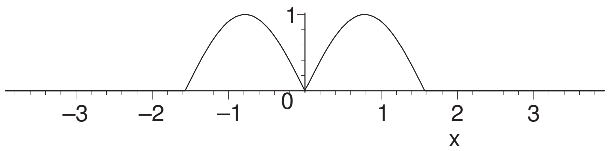
$$\frac{1}{2}a_0+a_1\cos t+\sum_{n=1}^{\infty}\{a_{4n-1}\cos(4n-1)t+a_{4n}\cos 4nt+a_{4n+1}\cos(4n+1)t\},$$

and find a_0 and a_1 and a_{4n-1}, a_{4n} and a_{4n+1}, $n\in\mathbb{N}$.

2) Prove that the Fourier series is uniformly convergent in $\mathbb{R}$.

3) Find for $t\in[-\pi,\pi]$ the sum of the series which is obtained by termwise integration from 0 to t of the Fourier series.

The function f is continuous and piecewise C^1 without vertical half tangents, so $f\in K_{2\pi}^*$. Then the Fourier series is by the **main theorem** convergent with the sum function $f^*(t)=f(t)$.



1) Now, f is *even*, so $b_n=0$, and

$$a_n=\frac{2}{\pi}\int_0^{\pi/2}\sin 2t\cdot\cos nt\,dt=\frac{1}{\pi}\int_0^{\pi/2}\{\sin(n+2)t-\sin(n-2)t\}dt.$$

We get for $n\neq 2$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left[-\frac{1}{n+2}\,\cos(n+2)t+\frac{1}{n-1}\,\cos(n-2)t\right]_0^{\pi/2}\\
&= \frac{1}{\pi}\left(-\frac{1}{n+2}\left\{\cos\left(\frac{n\pi}{2}+\pi\right)-1\right\}+\frac{1}{n-2}\left\{\cos\left(\frac{n\pi}{2}-\pi\right)-1\right\}\right)\\
&= \frac{1}{\pi}\left\{\frac{1}{n+2}\left(1+\cos\left(\frac{n\pi}{2}\right)\right)-\frac{1}{n-2}\left(1+\cos\left(\frac{n\pi}{2}\right)\right)\right\}\\
&= \frac{1}{\pi}\left(1+\cos\left(\frac{n\pi}{2}\right)\right)\cdot\frac{-4}{n^2-4}.
\end{aligned}$$

For $n=2$,

$$a_2=\frac{1}{\pi}\int_0^{\pi/2}\sin 4t\,dt=\left[-\frac{1}{4\pi}\,\cos 4t\right]_0^{\pi/2}=0.$$

Then

$$a_{4n+2} = -\frac{4}{(4n+2)^2-4}\cdot\frac{1}{\pi}(1+\cos\pi) = 0 \qquad \text{for } n\in\mathbb{N},$$

and it follows that the Fourier series has the right structure.

Then by a calculation,

$$a_0 = \frac{1}{\pi}(1+1)\cdot\frac{-4}{0^2-4} = \frac{2}{\pi}, \qquad\qquad a_1 = \frac{1}{\pi}\cdot(1+0)\cdot\frac{-4}{1^2-4} = \frac{4}{3\pi},$$

$$a_{4n-1} = \frac{1}{\pi}\left\{1+\cos\left(-\frac{\pi}{2}\right)\right\}\cdot\frac{-4}{(4n-1)^2-4} = -\frac{4}{\pi}\cdot\frac{1}{(4n-1)^2-4} = -\frac{4}{\pi}\cdot\frac{1}{(4n-3)(4n+1)},$$

$$a_{4n} = \frac{1}{\pi}(1+1)\cdot\frac{-4}{16n^2-4} = -\frac{2}{\pi}\cdot\frac{1}{4n^2-1} = -\frac{2}{\pi}\cdot\frac{1}{(2n-1)(2n+1)},$$

$$a_{4n+1} = -\frac{1}{\pi}\left\{1+\cos\left(\frac{\pi}{2}\right)\right\}\cdot\frac{-4}{(4n+1)^2-4} = -\frac{4}{\pi}\cdot\frac{1}{(4n+1)^2-4} = -\frac{4}{\pi}\cdot\frac{1}{(4n-1)(4n+3)}.$$

The Fourier series is (with equality sign, cf. the above)

$$f(t) = \frac{1}{\pi}+\frac{4}{3\pi}\cos t - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{2\cos(4n-1)t}{(4n-3)(4n-1)}+\frac{\cos 4nt}{(2n-1)(2n+1)}+\frac{2\cos(4n+1)t}{(4n-1)(4n+3)}\right\}.$$

2) Clearly, the Fourier series has a majoring series which is equivalent to the convergent series $c\sum_{n=1}^{\infty}\frac{1}{n^2}$. This implies that the Fourier series is absolutely and uniformly convergent.

3) The Fourier series being uniformly convergent, it can be termwise integrated for $x\in[-\pi,\pi]$,

$$\begin{aligned}\int_0^x f(t)\,dt \quad &= \quad \frac{x}{\pi}+\frac{4}{3\pi}\sin t - \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\frac{8\sin(4n-1)t}{(4n-3)(4n-1)(4n+1)}\right.\\ &\qquad\left.+\frac{\sin 4nt}{(2n-1)n(2n+1)}+\frac{8\sin(4n+1)t}{(4n-1)(4n+1)(4n+3)}\right\},\end{aligned}$$

where

$$\int_0^x f(t)\,dt = \begin{cases} 1 & \text{for } x\in\left[\frac{\pi}{2},\pi\right],\\[2mm] \frac{1}{2}(1-\cos 2x) & \text{for } x\in\left[0,\frac{\pi}{2}\right],\\[2mm] -\frac{1}{2}(1-\cos 2x) & \text{for } x\in\left[-\frac{\pi}{2},0\right],\\[2mm] -1 & \text{for } x\in\left[-\pi,-\frac{\pi}{2}\right].\end{cases}$$

Example 2.8 *Given the trigonometric series*

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2(n^2+1)}, \qquad x \in \mathbb{R}.$$

1) *Prove that it is pointwise convergent for every $x \in \mathbb{R}$. The sum function of the series is denoted by $g(x)$, $x \in \mathbb{R}$.*

2) *Prove that the trigonometric series*

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+1}$$

is uniformly convergent in $\mathbb{R}$.

3) *Find an expression of $g''(x)$ as a trigonometric series.*

It is given that the function f, $f \in K_{2\pi}$, given by

$$f(x) = \frac{x^2}{4} - \frac{\pi x}{2}, \qquad 0 \le x \le 2\pi,$$

has the Fourier series

$$-\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$

4) *Prove that g is that solution of the differential equation*

$$\frac{d^2y}{dx^2} - y = -f(x) - \frac{\pi^2}{6}, \qquad 0 \le x \le 2\pi,$$

for which $g'(0) = g'(\pi) = 0$, and find an expression as an elementary function of g for

$$0 \le x \le 2\pi.$$

5) *Find the exact value of*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

1) Since $\sum_{n=1}^{\infty} \dfrac{\cos(nx)}{n^2(n^2+1)}$ has the convergent majoring series

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n^2+1)},$$

the Fourier series is uniformly convergent and thus also pointwise convergent everywhere.

2) The series $\sum_{n=1}^{\infty} \dfrac{\cos(nx)}{n^2+1}$ has the convergent majoring series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

so it is also uniformly convergent.

3) Finally, the termwise differentiated series, $\sum_{n=1}^{\infty} \dfrac{-\sin(nx)}{n(n^2+1)}$ has the convergent majoring series

$$\sum_{n=1}^{\infty} \frac{1}{n(n^2+1)},$$

so it is also uniformly convergent. By another termwise differentiation we get from (2),

$$g''(x) = -\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+1}.$$

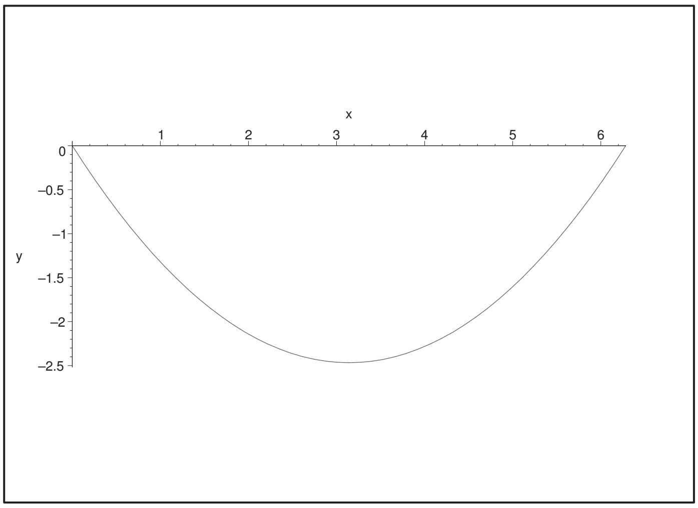
Intermezzo. We note that f is of class C^∞ in $]0,2\pi[$ and without vertical half tangents and that

$$f(0)=f(2\pi)=0.$$

Since the Fourier series for f by (2) is uniformly convergent, we have

$$f(x)=-\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2},$$

both pointwise and uniformly. The graph of f is shown on the figure.



4) The trigonometric series of g, g' and g'' are all uniformly convergent. When they are inserted into the differential equation, we get

$$\begin{aligned}\frac{d^2y}{dx^2}-y &= -\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1}-\sum_{n=1}^{\infty}\frac{\cos nx}{n^2(n^2+1)}=-\sum_{n=1}^{\infty}\frac{n^2+1}{n^2(n^2+1)}\cos nx\\ &= =-\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx=-f(x)-\frac{\pi^2}{6},\end{aligned}$$

and we have shown that they fulfil the differential equation.

Now,

$$g'(x)=-\sum_{n=1}^{\infty}\frac{\sin nx}{n(n^2+1)},$$

so $g'(0)=g'(\pi)=0$. It follows that g is a solution of the boundary value problem

$$(7)\quad \frac{d^2y}{dx^2}-y=-f(x)-\frac{\pi^2}{6}=-\frac{1}{4}\,x^2+\frac{\pi}{2}\,x-\frac{\pi^2}{6},$$

with the boundary conditions $y'(0)=y'(\pi)=0$ [notice, over half of the interval].

The corresponding homogeneous equation without the boundary conditions has the complete solution,

$$y = c_1 \cosh x + c_2 \sinh x.$$

Then we *guess* a particular solution of the form

$$y = ax^2 + bx + c.$$

When this is put into the left hand side of the equation we get

$$\frac{d^2y}{dx^2} - y = 2a - ax^2 - bx - c = -ax^2 - bx + (2a - c).$$

This equal to

$$-f(x) - \frac{\pi^2}{6} = -\frac{1}{4}\,x^2 + \frac{\pi}{2}\,x - \frac{\pi^2}{6},$$

if $a = \frac{1}{4}$, $b = -\frac{\pi}{2}$, $c - 2a = c - \frac{1}{2} = \frac{\pi^2}{6}$, thus $c = \frac{1}{2} + \frac{\pi^2}{6}$.

The complete solution of (7) is

$$y = \frac{1}{4}\,x^2 - \frac{\pi}{2}\,x + \frac{1}{2} + \frac{\pi^2}{6} + c_1 \cosh x + c_2 \sinh x.$$

Since

$$y' = \frac{1}{2}\,x - \frac{\pi}{2} + c_1 \sinh x + c_2 \cosh x,$$

it follows from the boundary conditions that

$$y'(0) = -\frac{\pi}{2} + c_2 = 0, \qquad \text{i.e. } c_2 = \frac{\pi}{2},$$

and

$$y'(\pi) = \frac{\pi}{2} - \frac{\pi}{2} + c_1 \sinh \pi + c_2 \cosh \pi = c_1 \sinh \pi + \frac{\pi}{2}\,\cosh \pi = 0,$$

hence $c_1 = -\frac{\pi}{2}\,\coth \pi$ and $c_2 = \frac{\pi}{2}$.

We see that the solution of the boundary value problem is unique. Since $g(x)$ is also a solution, we have obtained two expressions for $g(x)$, which must be equal,

$$(8)\quad g(x) = \frac{1}{4}\,x^2 - \frac{\pi}{2}\,x + \frac{1}{2} + \frac{\pi^2}{6} - \frac{\pi}{2}\coth \pi \cdot \cosh x + \frac{\pi}{2}\sinh x = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2(n^2+1)}.$$

5) Put $x = 0$ into (8). Then we get by a decomposition

$$g(0) = \frac{1}{2} + \frac{\pi^2}{6} - \frac{\pi}{2}\coth \pi = \sum_{n=1}^{\infty} \frac{1}{n^2(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2+1},$$

so by a rearrangement,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}\coth \pi - \frac{1}{2}.$$

Example 2.9 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = |t|^3 \qquad \text{for } -\pi < t \leq \pi.$$

1) Sketch the graph of f in the interval $[-\pi, \pi]$.

2) Prove that

$$f(t) = \frac{\pi^3}{4} + 6\pi \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi^2 n^4} \right\} \cos nt, \quad t \in \mathbb{R}.$$

Hint: We may use without proof that

$$\int_0^{\pi} t^3 \cos nt \, dt = 3\pi^2 \frac{(-1)^n}{n^2} + 6 \, \frac{1-(-1)^n}{n^4} \quad \text{for } n \in \mathbb{N}.$$

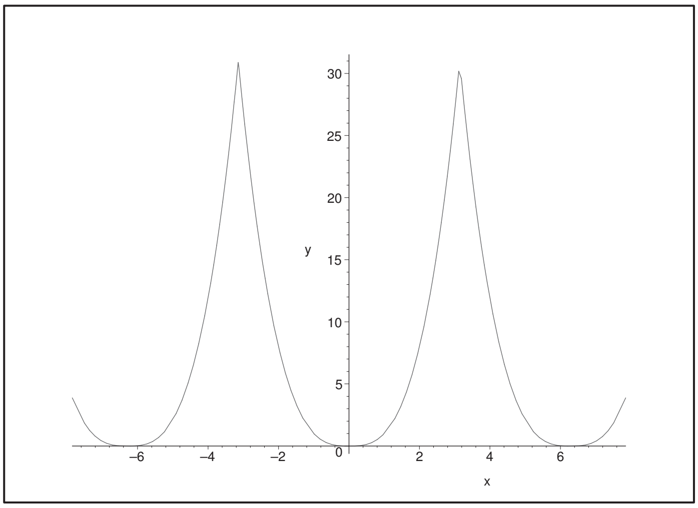
3) Prove that the Fourier series is uniformly convergent.

4) Apply the result of (2) to prove that

$$\sum_{p=1}^{\infty} \frac{1}{(2p-1)^4} = \frac{\pi^4}{96}.$$

Hint: Put $t = \pi$, and exploit that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

1) It follows from the graph that the function is continuous. It is clearly piecewise C^1 without vertical half tangents, so according to the **main theorem** the Fourier series for f is pointwise convergent with sum function $f(t)$, and we can even write $=$ instead of $\sim$.



2) Since $f(t)$ is even, all $b_n = 0$, and

$$a_n = \frac{2}{\pi}\int_0^{\pi} t^3 \cos nt\, dt, \qquad n \in \mathbb{N}_0.$$

For $n = 0$ (the exceptional case) we get

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t^3\, dt = \frac{\pi^4}{2\pi} = \frac{\pi^3}{2}.$$

When $n > 0$, we either use the hint, or partial integration. For completeness, the latter is shown below:

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} t^3 \cos nt\, dt = \frac{2}{\pi n}\left\{\left[t^3 \sin nt\right]_0^{\pi} - 3\int_0^{\pi} t^2 \sin nt\, dt\right\} \\
&= \frac{6}{\pi n^2}\left\{\left[t^2 \cos nt\right]_0^{\pi} - 2\int_0^{\pi} t \cos nt\, dt\right\} \\
&= \frac{6}{\pi n^2}\,\pi^2\cdot(-1)^n - \frac{12}{\pi n^3}\left\{[t\sin nt]_0^{\pi} - \int_0^{\pi} \sin nt\, dt\right\} \\
&= \frac{6\pi^2}{n^2}(-1)^n - \frac{12}{\pi n^4}[\cos nt]_0^{\pi} = 6\pi\left\{\frac{(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi^2 n^4}\right\}.
\end{aligned}$$

Then by insertion (remember $\frac{1}{2}\,a_0$) and application of the equality sign in stead of $\sim$ we therefore get

$$f(t) = \frac{\pi^3}{4} + 6\pi\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi^2 n^4}\right\}\cos nt, \quad t \in \mathbb{R}.$$

3) The Fourier series has clearly the majoring series

$$\frac{\pi^3}{4} + 6\pi \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{4}{\pi^2 n^4} \right) < \frac{\pi^3}{4} + 12\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This is convergent, so it follows that the Fourier series is uniformly convergent.

4) If we put $t = \pi$, then

$$\begin{aligned} f(\pi) = \pi^3 &= \frac{\pi^3}{4} + 6\pi \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n^2} + \frac{2(1-(-1)^n)}{\pi^2 n^4} \right\} (-1)^n \\ &= \frac{\pi^3}{4} + 6\pi \sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{12\pi}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4} \cdot (-1)^n \\ &= \frac{\pi^3}{4} + 6\pi \cdot \frac{\pi^2}{6} + \frac{12}{\pi} \sum_{p=1}^{\infty} \frac{2}{(2p-1)^4} (-1)^{2p-1} \\ &= \frac{\pi^3}{4} + \pi^3 - \frac{24}{\pi} \sum_{p=1}^{\infty} \frac{1}{(2p-1)^4}, \end{aligned}$$

hence by a rearrangement,

$$\sum_{p=1}^{\infty} \frac{1}{(2p-1)^4} = \frac{\pi}{24} \left(\frac{\pi^3}{4} + \pi^3 - \pi^3 \right) = \frac{\pi^4}{96}.$$

Example 2.10 *The periodic function $f : \mathbb{R} \to \mathbb{R}$ of period 2π is defined by*

$$f(t) = \begin{cases} 1, & t \in]-\pi, -\pi/2], \\ 0, & t \in]-\pi/2, \pi/2[, \\ 1, & t \in [\pi/2, \pi]. \end{cases}$$

Sketch the graph of f. Prove that f has the Fourier series

$$f \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos(2n-1)t.$$

Find the sum of the Fourier series and check if the Fourier series is uniformly convergent.

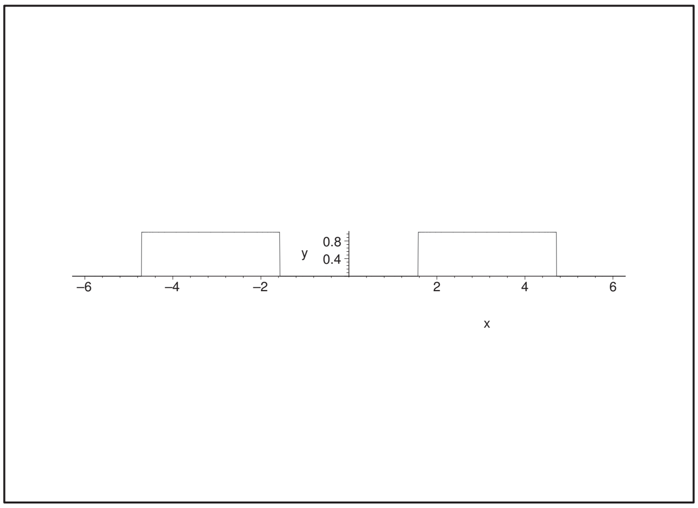
The function is even and piecewise C^1 without vertical half tangents, so $f \in K_{2\pi}^*$, and the Fourier series is a cosine series, $b_n = 0$. According to the **main theorem** we then have pointwise,

$$f^*(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

here the adjusted function $f^*(t)$ is given by

$$f^*(t) = \begin{cases} 1, & \text{for } t \in]-\pi, -\pi/2[, \\ 1/2, & \text{for } t = -\pi/2, \\ 0, & \text{for } t \in]-\pi/2, \pi/2[, \\ 1/2, & \text{for } t = \pi/2, \\ 1, & \text{for } t \in]\pi/2, \pi], \end{cases}$$

continued periodically, cf. figure.



The Fourier coefficients are

$$a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt = \frac{2}{\pi}\int_{\pi/2}^\pi \cos nt\,dt.$$

We get for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_{\pi/2}^\pi 1\,dt = 1, \qquad \text{thus} \quad \frac{1}{2}\,a_0 = \frac{1}{2}.$$

Then for $n \in \mathbb{N}$,

$$a_n = \frac{2}{\pi}\int_{\pi/2}^\pi \cos nt\,dt = \frac{2}{\pi n}[\sin nt]_{\pi/2}^\pi = -\frac{2}{\pi n}\,\sin n\,\frac{\pi}{2}.$$

If we split into the cases of n even or odd, we get

$$a_{2p} = -\frac{2}{\pi \cdot 2p}\cdot \sin p\pi = 0,$$

$$a_{2p-1} = -\frac{2}{\pi(2p-1)} \cdot \sin(2p-1)\frac{\pi}{2} = \frac{2}{\pi} \cdot \frac{(-1)^p}{2p-1}.$$

We get by insertion the given Fourier series (with equality sign for the adjusted function)

$$f^*(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^p}{2p-1} \cos(2p-1)t.$$

Since all terms $\cos(2p-1)t$ are continuous, and $f^*(t)$ [or $f(t)$ itself] is not, the convergence cannot be uniform.

Example 2.11 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = e^{|t|} \qquad \text{for } -\pi < t \le \pi.$$

1) Sketch the graph of f and explain why $f \in K_{2\pi}^*$.

2) Prove that the Fourier series for f is given by

$$\frac{1}{\pi}(e^{\pi}-1) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-e^{\pi}(-1)^n}{n^2+1} \cos nt.$$

3) Prove that the Fourier series is uniformly convergent.

1) Since f is piecewise C^{∞} without vertical half tangents, we have $f \in K_{2\pi}^*$. Now, f is continuous, cf. the figure, so it follows by the **main theorem** that $f(t)$ is pointwise equal its Fourier series. Since f is an even function, the Fourier series is a cosine series, thus

$$(9) \quad f(t) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

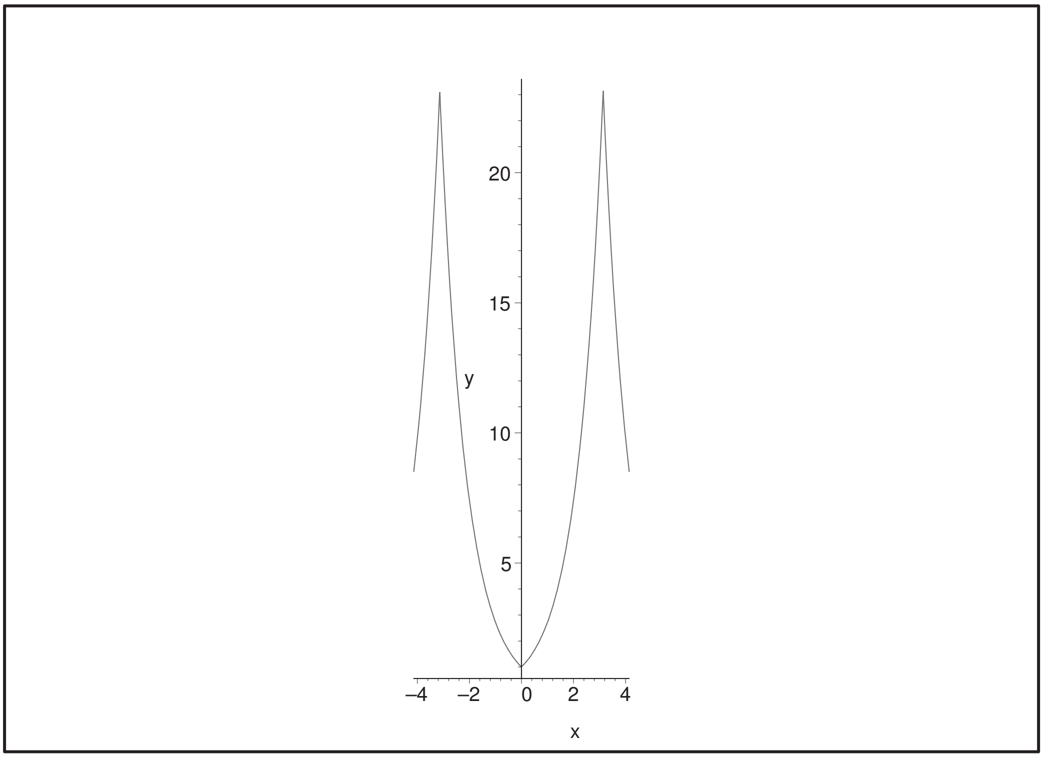
cf. the figure.

2) Then we get by successive partial integrations,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} e^t \cos nt\, dt = \frac{2}{\pi} \left[e^t \cos nt \right]_0^{\pi} + \frac{2n}{\pi} \int_0^{\pi} e^t \sin nt\, dt \\ &= \frac{2}{\pi} \left\{ (-1)^n e^{\pi} - 1 \right\} + \frac{2n}{\pi} \left[e^t \sin nt \right]_0^{\pi} - \frac{2n^2}{\pi} \int_0^{\pi} e^t \cos nt\, dt \\ &= \frac{2}{\pi}\{(-1)^n e^{\pi} - 1\} + 0 - n^2 a_n, \end{aligned}$$

so by a rearrangement,

$$a_n = \frac{2}{\pi} \cdot \frac{(-1)^n e^{\pi} - 1}{n^2+1}, \quad n \in \mathbb{N}_0, \quad \text{specielt } a_0 = \frac{2}{\pi}\{e^{\pi} - 1\}.$$



Alternatively it follows directly by complex calculations with $\cos nt = \operatorname{Re} e^{int}$ that

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^t \cos nt\,dt = \frac{2}{\pi}\operatorname{Re}\int_0^\pi e^{(1+in)t}dt = \frac{2}{\pi}\operatorname{Re}\left[\frac{1}{1+in}\,e^{(1+in)t}\right]_0^\pi \\ &= \frac{2}{\pi}\cdot\frac{1}{1+n^2}\operatorname{Re}[(1-in)\{e^\pi(-1)^n-1\}] = \frac{2}{\pi}\cdot\frac{(-1)^n e^\pi - 1}{n^2+1}. \end{aligned}$$

Then by insertion into (9) we get (pointwise equality by (1)) that

$$f(t) = \frac{e^\pi - 1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-e^\pi(-1)^n}{n^2+1}\cos nt.$$

3) The Fourier series has the convergent majoring series

$$\frac{e^\pi - 1}{\pi} + \frac{2}{\pi}(e^\pi+1)\sum_{n=1}^{\infty}\frac{1}{n^2+1} < \frac{e^\pi - 1}{\pi} + \frac{2}{\pi}(e^\pi+1)\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty,$$

so the Fourier series is uniformly convergent.

Example 2.12 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = (t-\pi)^2 \qquad \textit{for } \; -\pi < t \leq \pi.$$

1) Sketch the graph of f in the interval $[-\pi, \pi]$.

2) Prove that the Fourier series is convergent for every $t \in \mathbb{R}$, and sketch the graph of the sum function in the interval $[-\pi, \pi]$.

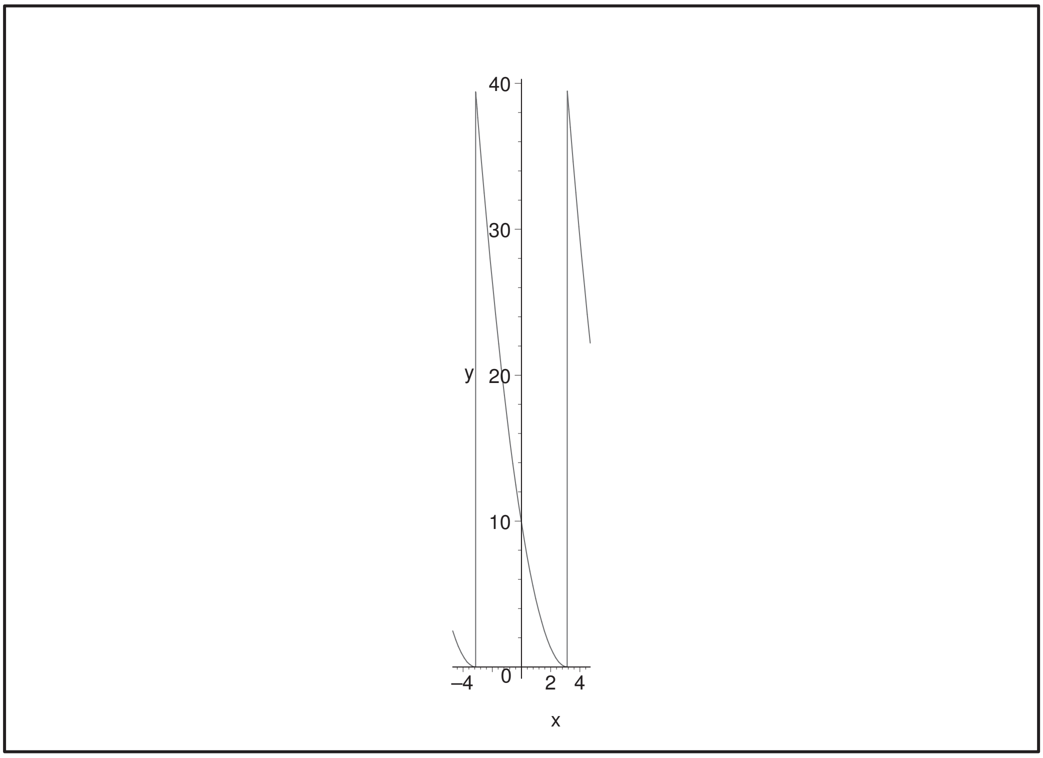
3) Explain why the Fourier series is not uniformly convergent.

4) Prove that the Fourier series for f is given by

$$\frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \left\{ \frac{(-1)^n}{n^2} \cos nt + \frac{(-1)^n \pi}{n} \sin nt \right\}, \qquad t \in \mathbb{R}.$$

(1) and (2) Since f is piecewise C^1 without vertical half tangents, we have $f \in K_{2\pi}^*$. Then by the **main theorem** the Fourier series is pointwise convergent with the adjusted function $f^*(t)$ as its sum function, where

$$f^*(t) = \begin{cases} f(t) & \text{for } t \neq (2p+1)\pi, \quad p \in \mathbb{Z}, \\ 2\pi^2 & \text{for } t = (2p+1)\pi, \quad p \in \mathbb{Z}. \end{cases}$$



(3) Since $f^*(t)$ is not continuous, the Fourier series cannot be uniformly convergent. In fact, if it was uniformly convergent, then the sum function should also be continuous, which it is not.

(4) We only miss the derivation of the Fourier series itself. For $n > 0$ we get by partial integration,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t-\pi)^2\cos nt\,dt = \frac{1}{\pi}\left[\frac{1}{n}\,\sin nt\cdot(t-\pi)^2\right]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}(t-\pi)\sin nt\,dt \\
&= 0 + \frac{2}{\pi n}\left[\frac{1}{n}\,\cos nt\cdot(t-\pi)\right]_{-\pi}^{\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi}\cos nt\,dt \\
&= \frac{2}{\pi n^2}\{0-(-1)^n\cdot(-2\pi)\} - 0 = \frac{4}{n^2}\,(-1)^n,
\end{aligned}$$

and for $n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(t-\pi)^2 dt = \frac{1}{\pi}\left[\frac{(t-\pi)^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\cdot\frac{8\pi^3}{3} = \frac{8\pi^2}{3},$$

and for $n \in \mathbb{N}$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t-\pi)^2\sin nt\,dt = \frac{1}{\pi}\left[-\frac{1}{\pi}\cos nt\cdot(t-\pi)^2\right]_{-\pi}^{\pi} + \frac{2}{\pi n}\int_{-\pi}^{\pi}(t-\pi)\cos nt\,dt \\
&= \frac{1}{\pi n}\cdot(-1)^n\cdot 4\pi^2 + \frac{2}{\pi n}[\sin nt\cdot(t-\pi)]_{-\pi}^{\pi} - \frac{2}{\pi n}\int_{-\pi}^{\pi}\sin nt\,dt = \frac{4\pi}{n}\cdot(-1)^n.
\end{aligned}$$

Summing up we get with pointwise equality,

$$f^*(t) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left\{\frac{(-1)^n}{n^2}\cos nt + \frac{(-1)^n\pi}{n}\sin nt\right\}, \quad t \in \mathbb{R}.$$

Example 2.13 *The periodic function $f : \mathbb{R} \to \mathbb{R}$ of period 2π is defined by*

$$f(t) = \begin{cases} \sin t, & t \in]-\pi, 0], \\ \\ \cos t, & t \in]0, \pi]. \end{cases}$$

It is given that f has the Fourier series

$$f \sim -\frac{1}{\pi} + \frac{\cos t + \sin t}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt + 2n\sin 2nt}{4n^2-1}.$$

1) Sketch the graph of for f.

2) Prove that the coefficients of $\cos nt$, $n \in \mathbb{N}_0$, in the Fourier series for f are as given above.

3) Find the sum function of the Fourier series and check if the Fourier series is uniformly convergent.

Let

$$f^+(t) = \frac{f(t)+f(-t)}{2}, \qquad f^-(t) = \frac{f(t)-f(-t)}{2}$$

be the even and the odd part of f, respectively.

4) Find the Fourier series for f^+, and check if it is uniformly convergent.

5) Find the Fourier series for f^-, and check if it is uniformly convergent.

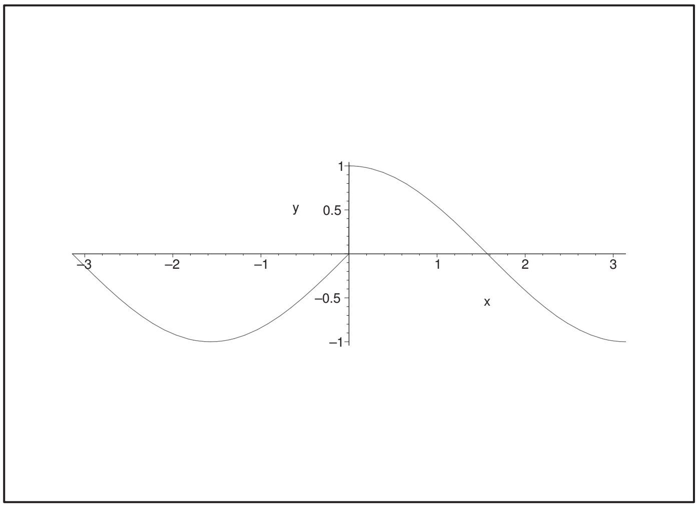
1) We note that f is piecewise differentiable without vertical half tangents. Then by the **main theorem** the Fourier series is pointwise convergent with the adjusted function $\tilde{f}$ as its sum function.

2) The coefficients a_n are defined by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}\sin t\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}\cos t\cos nt\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{0}\{\sin(n+1)t - \sin(n-1)t\}dt + \frac{1}{2\pi}\int_{0}^{\pi}\{\cos(n+1)t + \cos(n-1)t\}dt. \end{aligned}$$

In order to divide unawarely by 0, we immediately calculate separately the case $n = 1$:

$$a_1 = \frac{1}{2\pi}\int_{-\pi}^{0}\sin 2t\,dt + \frac{1}{2\pi}\int_{0}^{\pi}\{\cos 2t + 1\}dt = 0 + \frac{1}{2\pi}\{0+\pi\} = \frac{1}{2}.$$



Then we get for $n \geq 0$, $n \neq 1$,

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\left[-\frac{\cos(n+1)t}{n+1}+\frac{\cos(n-1)t}{n-1}\right]_{-\pi}^{0}+\frac{1}{2\pi}\left[\frac{\sin(n+1)t}{n+1}+\frac{\sin(n-1)t}{n-1}\right]_{0}^{\pi} \\
&= \frac{1}{2\pi}\left\{-\frac{1}{n+1}[1-(-1)^{n+1}]+\frac{1}{n-1}[1-(-1)^{n-1}]\right\} \\
&= \frac{1}{2\pi}\cdot\frac{2}{n^2-1}\{1+(-1)^n\}=\frac{1}{\pi}\cdot\frac{1}{n^2-1}\{1+(-1)^n\}.
\end{aligned}$$

It follows immediately, that if $n = 2p+1$, $p \in \mathbb{N}$, is odd and > 1, then

$$a_{2p+1} = 0.$$

When n is replaced by $2n$, we get

$$a_{2n} = \frac{1}{\pi}\cdot\frac{2}{4n^2-1}.$$

In particular we find for $n = 0$ that

$$\frac{1}{2}\,a_0 = -\frac{1}{\pi}.$$

Summing up we get

$$\frac{1}{2}\,a_0 = -\frac{1}{\pi}, \quad a_1 = \frac{1}{2}, \quad a_{2n} = \frac{1}{\pi}\cdot\frac{1}{4n^2-1}, \quad a_{2n+1} = 0, \quad \text{for } n \in \mathbb{N},$$

in agreement with the given Fourier series.

3) According to (1) we have pointwise convergence with the sum

$$\tilde{f}(t) = \begin{cases} \sin t, & t \in]-\pi, 0[, \\[2mm] 1/2, & t = 0, \\ \cos t, & t \in]0, \pi[, \\[2mm] -1/2, & t = \pi, \end{cases}$$

which is continued periodically.

Since f (and $\tilde{f}$) is not continuous, the Fourier series for f cannot be uniformly convergent.

4) The Fourier series for f^+ is the even part of the Fourier series for f, thus

$$f^+ \sim -\frac{1}{\pi} + \frac{1}{2}\cos t + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nt.$$

This is clearly uniformly convergent, because it has the convergent majoring series

$$\frac{1}{\pi} + \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \le 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{\pi}{3}.$$

Remark 2.2 It follows that

$$\tilde{f}^+(t) = \begin{cases} \{\sin t + \cos t\}/2 & \text{for } t \in]-\pi, 0[, \\ 1/2 & \text{for } t = 0, \\ \{-\sin t + \cos t\}/2 & \text{for } t \in]0, \pi[, \\ -1/2 & \text{for } t = \pi, \end{cases}$$

hence the continuation of f^+ is continuous and piecewise C^1 without vertical half tangents.

5) The Fourier series for f^- is the odd part of the Fourier series for f, thus

$$f^- \sim \frac{1}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\,\sin 2nt.$$

If this series was uniformly convergent, then f^- should be continuous, and hence also $f = f^+ + f^-$ continuous. ¡vspace3mm

However, f is *not* continuous, so the Fourier series for f^- is not uniformly convergent.

Example 2.14 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = \frac{1}{4\pi^2}\,t(4\pi - t), \qquad t \in [0, 2\pi[.$$

1) Sketch the graph of f in the interval $[-2\pi, 2\pi[$.

2) Explain why the Fourier series is pointwise convergent for every $t \in \mathbb{R}$, and sketch the graph of the sum function in the interval $[-2\pi, 2\pi[$.

3) Show that the Fourier series is not uniformly convergent.

4) Prove that the Fourier series for f is given by

$$\frac{2}{3} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\cos nt + \frac{\pi}{n}\sin nt\right\}, \qquad t \in \mathbb{R}.$$

Hint: One may use without proof that

$$\int t(4\pi - t)\cos nt\,dt = \frac{2}{n^2}(2\pi - t)\cos nt + \left\{\frac{2}{n^3} + \frac{t(4\pi - t)}{n}\right\}\sin nt,$$

and

$$\int t(4\pi - t)\sin nt\,dt = \frac{2}{n^2}(2\pi - t)\sin nt - \left\{\frac{2}{n^3} + \frac{t(4\pi - t)}{n}\right\}\cos nt,$$

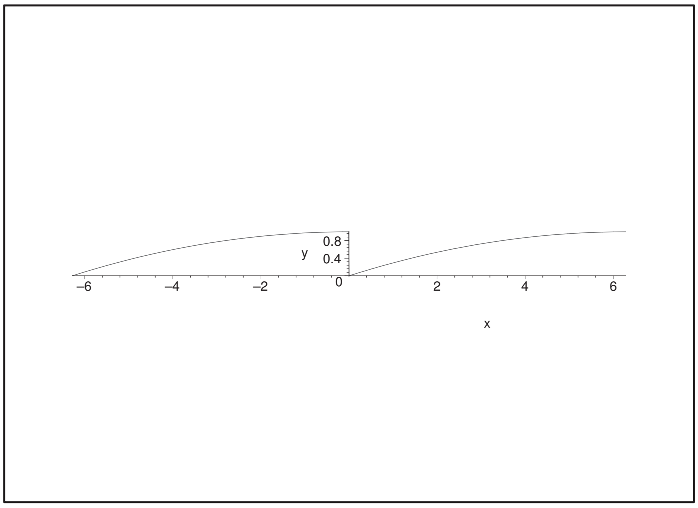
for $n \in \mathbb{N}$.

(1) and (2) It follows from the rearrangement,

$$f(t) = \frac{1}{4\pi^2}\{4\pi^2 - (t - 2\pi)^2\}$$

that the graph of $f(t)$ in $[0, 2\pi[$ is a part of an arc of a parabola with its vertex at $(2\pi, 1)$. The normalized function $f^*(t)$ is equal to $\frac{1}{2}$ for $t = 2p\pi$, $p \in \mathbb{Z}$, and $= f(t)$ at any other point. Since $f(t)$ is of class C^∞ in $]0, 2\pi[$ and without vertical half tangents, the Fourier series is by the **main theorem** pointwise convergent with $f^*(t)$ as its sum function.

(3) Since $f^*(t)$ is *not* continuous, it follows that the Fourier series *cannot* be uniformly convergent.



(4) We are now only missing the Fourier coefficients. We calculate there here without using the hints above:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4\pi^2}(4\pi t - t^2)dt = \frac{1}{4\pi^3}\left[2\pi t^2 - \frac{1}{3}\,t^3\right]_0^{2\pi} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}.$$

We get for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \int \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4\pi^2}(4\pi t - t^2)\cos nt\,dt \\ &= \frac{1}{4\pi^3}\left[\frac{1}{n}\,t(4\pi - t)\sin nt\right]_0^{2\pi} - \frac{1}{4\pi^3 n}\int_0^{2\pi}(4\pi - 2t)\sin nt\,dt \\ &= \frac{2}{4\pi^3 n^2}[(2\pi - t)\cos nt]_0^{2\pi} + \frac{1}{2\pi^2 n^2}\int_0^{2\pi}\cos nt\,dt = -\frac{2\pi}{2\pi^3 n^2} = -\frac{1}{\pi^2}\cdot\frac{1}{n^2}, \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4\pi^2}(4\pi t - t^2)\sin nt\,dt \\ &= \frac{1}{4\pi^3}\left[-\frac{1}{n}\,t(4\pi - t)\cos nt\right]_0^{2\pi} + \frac{1}{2\pi^3 n}\int_0^{2\pi}(2\pi - t)\cos nt\,dt \\ &= \frac{1}{4\pi^3 n}(-2\pi\cdot 2\pi) + \frac{1}{2\pi^3 n^2}[(2\pi - t)\sin nt]_0^{2\pi} + \frac{1}{2\pi^3 n^2}\int_0^{2\pi}\sin nt\,dt = -\frac{\pi}{\pi^2 n}. \end{aligned}$$

Hence we get the Fourier series with its sum function $f^*(t)$,

$$\begin{aligned} f^*(t) &= \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty}\{a_n\cos nt + b_n \sin nt\} \\ &= \frac{2}{3} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2}\,\cos nt + \frac{\pi}{n}\,\sin nt\right\}, \qquad t \in \mathbb{R}. \end{aligned}$$

Example 2.15 *We define for every fixed r, $0 < r < 1$, the function $f_r : \mathbb{R} \to \mathbb{R}$, by*

$$f_r(t) = \ln(1 + r^2 - 2r\cos t), \qquad t \in \mathbb{R}.$$

1) Explain why $f_r \in K^*_{2\pi}$.

Prove that f_r has the Fourier series

$$(10) \qquad -2\sum_{n=1}^{\infty} \frac{r^2}{n}\cos(nt).$$

2) Prove that the Fourier series (10) is uniformly convergent for $t \in \mathbb{R}$, and find its sum function.

3) Calculate the value of each of the integrals

$$\int_0^{2\pi} f_r(t)\,dt, \quad \int_0^{2\pi} f_r(t)\cos(5t)\,dt, \quad \int_{-\pi}^{\pi} f_r(t)\sin(5t)\,dt.$$

4) Find the sum of each of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n} \quad og \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{3^n \cdot n}.$$

5) Prove that the series which is obtained by termwise differentiation (with respect to t) of (10), is uniformly convergent for $t \in \mathbb{R}$, and find the sum of the differentiated series in $-\pi \le t \le \pi$.

1) Clearly, $f_r(t)$ is defined and C^∞ in t, when $0 < r < 1$, because we have

$$1 + r^2 - 2r\cos t \ge 1 + r^2 - 2r = (1-r)^2 > 0.$$

Since it is also periodic of period 2π, it follows that $f_r \in K^*_{2\pi}$. Then by the **main theorem** the Fourier series for each $f_r(t)$, $0 < r < 1$, is pointwise convergent with $f_r(t)$ as its sum function.

Then we prove that the Fourier series becomes

$$f_r(t) = \ln(1 + r^2 - 2r\cos t) = -2\sum_{n=1}^{\infty} \frac{r^n}{n}\cos(nt), \qquad 0 < r < 1,$$

where we have earlier noted that the equality sign is valid.

First note that the quotient series expansion

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad \text{for } |z| < 1,$$

also holds for complex $z \in \mathbb{C}$, if only $|z| < 1$.

Then put $z = re^{it}$, thus $|z| = r \in]0,1[$, and we get by Moivre's formula that

$$z^n = r^n e^{int} = r^n\{\cos nt + i\sin nt\}.$$

Then we get for $0 < r < 1$ by insertion into the quotient series that

$$\frac{1}{1-z} = \frac{1}{1-re^{it}} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} r^n\{\cos nt + i \sin nt\}.$$

Here we take two times the imaginary part,

$$\begin{aligned} 2\sum_{n=1}^{\infty} r^n \sin nt &= 2\,\mathrm{Im}\left(\frac{1}{1-re^{it}}\right) = 2\,\mathrm{Im}\left(\frac{1}{1-re^{it}} \cdot \frac{1-re^{-it}}{1-re^{-it}}\right) \\ &= \frac{2r\sin t}{1+r^2-r(e^{it}+e^{-it})} = \frac{2r\sin t}{1+r^2-2r\cos t}. \end{aligned}$$

The series has the convergent majoring series

$$2\sum_{n=1}^{\infty} r^n = \frac{2r}{1-r} < \infty,$$

so it is uniformly convergent. We may therefore perform termwise integration,

$$-2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos nt = \ln(1+r^2-2r\cos t)+c,$$

where the constant c is fixed by putting $t=0$ and then apply the logarithmic series,

$$\begin{aligned}\ln(1+r^2-2r)+c &= \ln\{(1-r)^2\}+c = 2\ln(1-r)+c\\ &= -2\sum_{n=1}^{\infty}\frac{r^n}{n} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-r)^n = 2\ln(1-r).\end{aligned}$$

We get $c=0$, and we have proved that we have uniformly that

$$\ln(1+r^2-2r\cos t) = -2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos nt.$$

This intermezzo contains latently (2) and (5); but we shall not use this fact here.

2) If $0<r<1$ is kept fixed, then we have the trivial estimate

$$\left|-2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos nt\right| \le 2\sum_{n=1}^{\infty}\frac{r^n}{n} = 2\ln\frac{1}{1-r} < \infty.$$

The Fourier series has a convergent majoring series, so it is uniformly convergent.

3) This question may be answered in many different ways.

a) **First variant**. It follows from the definition of a Fourier series that

$$f_r(t) \sim -2\sum_{n=1}^{\infty}\frac{r^n}{n}\cos nt = \frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n\cos nt+b_n\sin nt\},$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f_r(t)\cos nt\,dt,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f_r(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f_r(t)\sin nt\,dt.$$

Then by identification,

$$\int_0^{2\pi} f_r(t)\,dt = \pi a_0 = 0,$$

$$\int_0^{2\pi} f_r(t)\cos(5t)dt = \pi a_5 = -\frac{2\pi r^5}{5},$$

$$\int_{-\pi}^{\pi} f_r(t)\sin(5t)dt = \pi b_5 = 0.$$

b) **Second variant**. Since the series expansion

$$f_r(t) = -2\sum_{n=1}^{\infty} \frac{r^n}{n} \cos(nt), \qquad 0 < r < 1,$$

is uniformly convergent, it follows by interchanging the summation and integration that

$$\int_0^{2\pi} f_r(t)\,dt = -2\sum_{n=1}^{\infty} \frac{r^n}{n} \int_0^{2\pi} \cos nt\,dt = 0,$$

$$\begin{aligned}\int_0^{2\pi} f_r(t)\cos(5t)\,dt &= -2\sum_{n=1}^{\infty} \frac{r^n}{n} \int_0^{2\pi} \cos nt \cdot \cos(5t)dt \\ &= -2\cdot\frac{r^5}{5}\int_0^{2\pi} \cos^2(5t)dt = -\frac{2\pi r^5}{5},\end{aligned}$$

$$\int_{-\pi}^{\pi} f_r(t)\sin(5t)dt = -2\sum_{n=1}^{\infty} \frac{r^n}{n} \int_{-\pi}^{\pi} \cos nt \cdot \sin 5t\,dt = 0.$$

Remark 2.3 A direct integration of e.g.

$$\int_0^{2\pi} f_r(t)\cos(5t)dt = \int_0^{2\pi} \ln(1+r^2-2r\cos t)\cdot\cos(5t)dt$$

does not look promising and my pocket calculator does not either like this integral.

4) Here, we also have two variants.

a) **First variant.** Since

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos nt = -\frac{1}{2} f_r(t) = -\frac{1}{2}\ln(1+r^2-2r\cos t),$$

we get by choosing $r = \frac{1}{2}$ and $t = 0$ that

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = -\frac{1}{2} f_{1/2}(0) = -\frac{1}{2}\ln\left(1+\frac{1}{4}-2\cdot\frac{1}{2}\right) = -\frac{1}{2}\ln\left(\frac{1}{4}\right) = \ln 2.$$

If we instead choose $r = \frac{1}{3}$ and $t = \pi$, we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} = -\frac{1}{2} f_{1/3}(\pi) = -\frac{1}{2}\ln\left(1+\frac{1}{9}+\frac{2}{3}\right) = -\frac{1}{2}\ln\frac{16}{9} = \ln\frac{3}{4}.$$

b) **Second variant**. If we instead use the series expansion

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^n}{n} = \ln(1+r), \qquad r \in\,]-1,1[,$$

we obtain for $r = -\dfrac{1}{2}$ that

$$-\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln\left(1-\frac{1}{2}\right) = -\ln 2, \qquad \text{dvs.} \quad \sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2.$$

Then for $r = \dfrac{1}{3}$,

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} = \ln\left(1+\frac{1}{3}\right) = \ln\frac{4}{3}, \qquad \text{dvs.} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n} = \ln\frac{3}{4}.$$

5) When we perform termwise differentiation of the Fourier series. we get

$$2\sum_{n=1}^{\infty} r^n \sin nt.$$

If $0 < r < 1$, then $2\sum_{n=1}^{\infty} r^n = \dfrac{2r}{1-r} < \infty$ is a convergent majoring series. Consequently, the differentiated series is uniformly convergent with the sum function $f_r'(t)$, thus

$$2\sum_{n=1}^{\infty} r^n \sin nt = \frac{d}{dt}\ln(1+r^2-2r\cos t) = \frac{2t\sin t}{1+r^2-2r\cos t}.$$

3 Parseval's equation

Example 3.1 *A function $f \in K_{2\pi}$ is given in the interval $]-\pi,\pi]$ by*

$$f(t) = \begin{cases} \dfrac{2\pi}{3} - |t|, & \text{for } |t| \le \dfrac{2\pi}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

1) Sketch the graph of f in the interval $]\pi,\pi]$.
Prove that f has the Fourier series

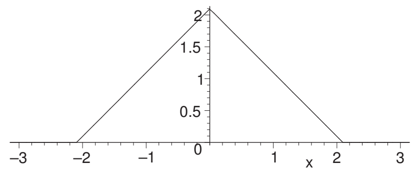
$$\frac{2\pi}{9} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left(1 - \cos\left(n\,\frac{2\pi}{3}\right)\right)\cos nt, \qquad t \in \mathbb{R}.$$

2) Given that

$$\int_{\pi}^{\pi} f(t)^2 dt = \frac{16}{81}\,\pi^3,$$

find the sum of the series

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{11^4} + \cdots .$$



1) Since f is continuous and piecewise C^1 without vertical half tangents, we have $f \in K^*_{2\pi}$. The Fourier series is then by the **main theorem** pointwise convergent and its sum function is $f^*(t) = f(t)$, because $f(t)$ is continuous. Now, $f(t)$ is even, so $b_n = 0$, and

$$a_0 = \frac{2}{\pi}\int_0^{2\pi/3}\left(\frac{2\pi}{3} - t\right)dt = -\frac{1}{\pi}\left[\left(\frac{2\pi}{3} - t\right)^2\right]_0^{2\pi/3} = \frac{1}{\pi}\left(\frac{2\pi}{3}\right)^2 = \frac{4\pi}{9}.$$

We get for $n > 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{2\pi/3}\left(\frac{2\pi}{3} - t\right)\cos nt\,dt = \frac{2}{\pi}\left[\frac{1}{n}\left(\frac{2\pi}{3} - t\right)\sin nt\right]_0^{2\pi/3} + \frac{2}{\pi n}\int_0^{2\pi/3}\sin nt\,dt \\ &= \frac{2}{\pi n^2}[-\cos nt]_0^{2\pi/3} = \frac{2}{\pi n^2}\left(1 - \cos\left(\frac{2\pi}{3}\,n\right)\right). \end{aligned}$$

The Fourier series is then with an equality sign, cf. the above,

$$f(t) = \frac{2\pi}{9} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left\{1 - \cos\left(n\,\frac{2\pi}{3}\right)\right\}\cos nt.$$

2) By **Parseval's equation** we get

$$(11)\quad \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{2}\,a_0^2 + \sum_{n=1}^{\infty} a_n^2 = \frac{1}{2}\left(\frac{4\pi}{9}\right)^2 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left\{1-\cos\left(\frac{2\pi n}{3}\right)\right\}^2 \cdot \frac{1}{n^4}.$$

Here

$$\int_{-\pi}^{\pi} f(t)^2 dt = 2\int_0^{2\pi/3}\left(\frac{2\pi}{3}-t\right)^2 dt = -\frac{2}{3}\left[\left(\frac{2\pi}{3}-t\right)^3\right]_0^{2\pi/3} = \frac{2}{3}\left(\frac{2\pi}{3}\right)^3 = \frac{16\pi^3}{81},$$

and

$$\left\{1-\cos\left(\frac{2\pi n}{3}\right)\right\}^2 = \left\{2\sin^3\left(\frac{\pi n}{n}\right)\right\}^2 = 4\sin^4\left(n\,\frac{\pi}{3}\right) = \begin{cases} 4\left(\pm\dfrac{\sqrt{3}}{2}\right)^4 = \dfrac{9}{4} & n \neq 3p, \\ 0 & n = 3p. \end{cases}$$

Then by insertion into (11),

$$\frac{16\pi^2}{81} = \frac{8\pi^2}{81} + \frac{4}{\pi^2}\cdot\frac{9}{4}\left\{\sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n=1}^{\infty}\frac{1}{(3n)^4}\right\},$$

hence by a rearrangement,

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{8^4}+\cdots$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n_1}^{\infty}\frac{1}{(3n)^4} = \left(1-\frac{1}{81}\right)\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{80}{81}\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^2}{9}\cdot\frac{8\pi^2}{81} = \frac{8\pi^4}{729}.$$

Note that it follows from the above that

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{81}{80}\cdot\frac{8\pi^4}{729} = \frac{\pi^4}{90}.$$

Example 3.2 *A periodic function $f:\mathbb{R}\to\mathbb{R}$ of period 2π is given in the interval $]-\pi,\pi]$ by*

$$f(t)=\left\{\pi|t|-t^2\right\}^2,\qquad t\in\,]-\pi,\pi].$$

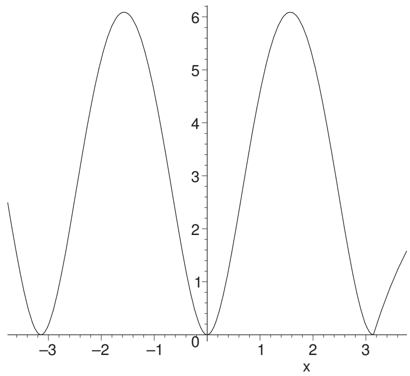
1) Prove that f has the Fourier series

$$\frac{\pi^4}{30}-\sum_{n=1}^{\infty}\frac{3}{n^4}\cos 2nt.$$

2) Find the sum of the series $\sum_{n=1}^{\infty}\dfrac{1}{n^8}$.

3) Prove that the series which is obtained by termwise differentiation of the Fourier series above is uniformly convergent in $\mathbb{R}$. Find the sum of the termwise differentiated series for $t\in\,]-\pi,\pi]$.

The function f is continuous and piecewise C^1 without vertical half tangents, hence $f\in K_{2\pi}^*$. The Fourier series is by the **main theorem** pointwise convergent and its sum function is $f^*(t)=f(t)$.



1) Since $\pi|t|-t^2$ is *even*, $f(t)$ is also even, and the series is a cosine series, hence $b_n=0$, and

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} f(t)\,dt=\frac{2}{\pi}\int_0^{\pi}\left(\pi t-t^2\right)^2 dt\\
&= \frac{2}{\pi}\int_0^{\pi}\left(t^4-2\pi t^3+\pi^2t^2\right)dt=\frac{2}{\pi}\left(\frac{\pi^5}{5}-\frac{1}{2}\pi^5+\frac{1}{3}\pi^5\right)\\
&= \frac{2\pi^4}{30}(6-15+10)=\frac{\pi^4}{15}.
\end{aligned}$$

If $n\in\mathbb{N}$, we get instead

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi}(t^4-2\pi t^3+\pi^2t^2)\cos nt\,dt\\
&= \frac{2}{\pi n}\left[\left(t^2-\pi t\right)^2\sin nt\right]_0^{\pi}-\frac{2}{\pi n}\int_0^{\pi}(4t^3-6\pi t^2+2\pi^2 t)\sin nt\,dt
\end{aligned}$$

so

$$
\begin{aligned}
a_n &= 0+\frac{4}{\pi n^2}\left[\left(2t^3-3\pi t^2+\pi^2 t^2\right)\right]_0^{\pi}-\frac{4}{\pi n^2}\int_0^{\pi}\left(6t^2-6\pi t+\pi^2\right)\cos nt,dt \\
&= 0-\frac{4}{\pi n^3}\left[\left(6t^2-6\pi t+\pi^2\right)\sin nt\right]_0^{\pi}+\frac{24}{n^3\pi}\int_0^{\pi}(2t-\pi)\sin nt\,dt \\
&= 0-\frac{24}{\pi n^4}[(2t-\pi)\cos nt]_0^{\pi}+\frac{48}{\pi n^4}\int_0^{\pi}\cos nt\,dt \\
&= -\frac{24}{\pi n^4}\left\{\pi(-1)^n+\pi\right\}=-\frac{24}{n^4}\left\{1+(-1)^n\right\}.
\end{aligned}
$$

Thus we get $a_{2n+1}=0$ for $n\geq 0$, and

$$a_{2n}=-\frac{24}{(2n)^4}\cdot 2=-\frac{3}{n^4}\qquad \text{for } n\in\mathbb{N}.$$

Summing up the Fourier series is with equality sign (by the beginning of the example),

$$f(t)=\left\{\pi|t|-t^2\right\}^2=\frac{\pi^4}{30}-\sum_{n=1}^{\infty}\frac{3}{n^4}\cos 2nt,\qquad t\in[-\pi,\pi].$$

2) We get by **Parseval's equation** that

$$\frac{1}{2}\,a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2dt.$$

In the present case,

$$\frac{1}{2}\,a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{2}\left\{\frac{\pi^4}{15}\right\}^2+9\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{450}+9\sum_{n=1}^{\infty}\frac{1}{n^8},$$

and

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2dt &= \frac{2}{\pi}\int_0^{\pi} t^4(\pi-t)^4dt=\frac{2}{\pi}\left[\frac{t^5}{5}\,(\pi-t)^4\right]_0^{\pi}+\frac{2}{\pi}\cdot\frac{4}{5}\int_0^{\pi} t^5(\pi-t)^3dt\\
&= 0+\frac{2}{\pi}\cdot\frac{4}{5}\left[\frac{1}{6}\,t^6(\pi-t)^3\right]_0^{\pi}+\frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\int_0^{\pi} t^6(\pi-t)^2dt\\
&= 0+\frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\left[\frac{1}{7}\,t^7(\pi-t)^2\right]_0^{\pi}+\frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\cdot\frac{2}{7}\int_0^{\pi} t^7(\pi-t)dt\\
&= 0+\frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\cdot\frac{2}{7}\left[\frac{1}{8}\,t^8(\pi-t)\right]_0^{\pi}+\frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\cdot\frac{2}{7}\cdot\frac{1}{8}\int_0^{\pi} t^8dt\\
&= \frac{2}{\pi}\cdot\frac{4}{5}\cdot\frac{3}{6}\cdot\frac{2}{7}\cdot\frac{1}{8}\cdot\frac{\pi^9}{9}=\frac{\pi^8}{5\cdot 7\cdot 9}=\frac{\pi^8}{315}.
\end{aligned}$$

By insertion into **Parseval's equation** we get

$$\frac{\pi^8}{450}+9\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{315},$$

hence by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9}\left(\frac{1}{315}-\frac{1}{450}\right)=\frac{135}{315\cdot 450}\cdot\frac{\pi^8}{9}=\frac{9\cdot 15}{3\cdot 105\cdot 3\cdot 150}\cdot\frac{\pi^8}{9}=\frac{\pi^8}{9450}.$$

3) The termwise differentiated series

$$6\sum_{n=1}^{\infty}\frac{1}{n^3}\,\sin 2nt$$

has the convergent majoring series $\sum_{n=1}^{\infty}\dfrac{6}{n^3}$, hence the series is uniformly convergent.

Its sum is for $t\in\,]0,\pi[$ given by

$$f'(t)=\frac{d}{dt}\,\left(\pi t-t^2\right)^2=\frac{d}{dt}\left(t^4-2\pi t^3+\pi^2t^2\right)=4t^3-6\pi t^2+2\pi^2 t.$$

Analogously, we get for $t\in\,]-\pi,0[$,

$$f'(t)=4t^3+6\pi t^2+2\pi t.$$

Then by a continuous continuation, $f'(\pi) = f'(-\pi) = 0$ and $f'(0) = 0$, so

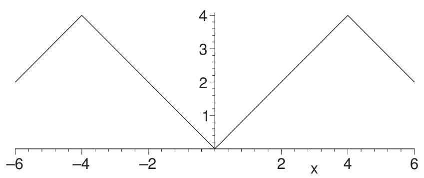
$$6\sum_{n=1}^{\infty} \frac{1}{n^3}\,\sin 2nt = \begin{cases} 4t^3 + 6\pi t^2 + 2\pi^2 t & \text{for } t \in [-\pi, 0], \\ 4t^3 - 6\pi t^2 + 2\pi^2 t & \text{for } t \in [0, \pi]. \end{cases}$$

Example 3.3 *Find the Fourier series for the function $f \in K_8$, which is given in the interval $]-4, 4]$ by*

$$f(t) = \begin{cases} -t & \text{for } -4 < t \le 0, \\ t & \text{for } 0 < t \le 4. \end{cases}$$

Then apply Parseval's equation in order to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}.$$



As in Example 1.7 we see that $f \in K_8^*$. The function f is continuous and *even*, so it follows from the **main theorem** that the symbol $\sim$ can be replaced by an equality sign. Furthermore, the series is a cosine series,

$$f(t) = \frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{4},$$

where we have for $n \in \mathbb{N}$,

$$\begin{aligned} a_n &= \frac{4}{8}\int_0^4 t\cos\left(\frac{2\pi n}{8}\,t\right)dt = \frac{1}{2}\int_0^4 t\cos\left(\frac{n\pi t}{4}\right)dt = \frac{1}{2}\left[\frac{4}{n\pi}\,t\cdot\sin\left(\frac{n\pi t}{4}\right)\right]_0^4 - \frac{2}{n\pi}\int_0^4 \sin\left(\frac{n\pi t}{t}\right)dt \\ &= 0 + \frac{8}{n^2\pi^2}\left[\cos\left(\frac{n\pi t}{4}\right)\right]_0^4 = \frac{8}{n^2\pi^2}\left\{(-1)^n - 1\right\}. \end{aligned}$$

For $n = 0$ we get instead

$$a_0 = \frac{1}{2}\int_0^4 t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^4 = 4.$$

Since

$$(-1)^n - 1 = \begin{cases} 0 & \text{for } n \text{ even}, \\ -2 & \text{for } n \text{ odd}, \end{cases}$$

we get $a_{2n} = 0$ for $n \in \mathbb{N}$ (however, $a_0 = 4$ for $n = 0$), and

$$a_{2n-1} = -\frac{16}{\pi^2} \cdot \frac{1}{(2n-1)^2}, \qquad \text{for } n \in \mathbb{N}.$$

Summing up we get the Fourier series with equality sign instead of the symbol $\sim$)

$$f(t) = 2 - \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\frac{\pi}{4}\, t.$$

Whenever **Parseval's equation** is applied, it is always a good strategy first to identify all coefficients. In particular, we must be very careful with a_0, because it due to the factor $\frac{1}{2}$ plays a special role:

$$a_0 = 4, \quad a_{2n-1} = -\frac{16}{\pi^2} \cdot \frac{1}{(2n-1)^2}, \quad a_{2n} = 0, \quad b_n = 0 \quad \text{for } n \in \mathbb{N}.$$

Then by **Parseval's equation**,

$$\frac{1}{2}\,a_0^2+\sum_{n=1}^{\infty}\{a_n^2+b_n^2\}=\frac{2}{8}\int_{-4}^{4}t^2\,dt=\frac{1}{2}\int_{0}^{4}t^2\,dt=\frac{1}{2}\left[\frac{t^3}{3}\right]_0^4=\frac{32}{3},$$

hence

$$\frac{32}{3}=\frac{1}{2}\,a_0^2+\sum_{n=1}^{\infty}a_{2n-1}^2=\frac{16}{2}+\frac{16^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4},$$

and we get by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{16^2}\left\{\frac{32}{3}-\frac{16}{2}\right\}=\frac{\pi^4}{16}\left\{\frac{2}{3}-\frac{1}{2}\right\}=\frac{\pi^4}{16}\cdot\frac{1}{6}=\frac{\pi^4}{96}.$$

Example 3.4 *A periodic function $f:\mathbb{R}\to\mathbb{R}$ of period 2π is defined by*

$$f(t)=\begin{cases}\cos 2qt, & \text{for } t\in[0,\pi],\\[2mm] 0, & \text{for } t\in\,]-\pi,0[,\end{cases}$$

where $q\in\mathbb{N}$ is a constant.

1) Find the Fourier series of the function.

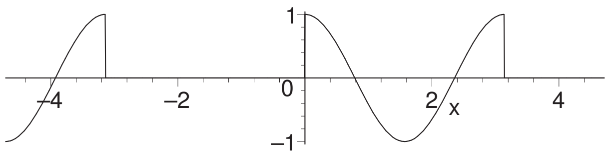
2) Use the Fourier series to find the sum of the series

$$\sum_{p=1}^{\infty}\frac{(2p-1)^2}{\left[(2p-1)^2-4q^2\right]^2}$$

er $\dfrac{1}{16}\,\pi^2$.

The function f is piecewise C^1 without vertical half tangents, so $f\in K_{2\pi}^*$. The Fourier series is then by the **main theorem** convergent, and its sum function is

$$f^*(t)=\begin{cases}\dfrac{1}{2} & \text{for } t=p\pi,\ p\in\mathbb{Z},\\[2mm] f(t) & \text{ellers.}\end{cases}$$



1) Now,

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos 2qt\, \cos nt\, dt = \frac{1}{2\pi}\int_0^{\pi}\{\cos(2q+n)t + \cos(2q-n)t\}dt,$$

so $a_n = 0$ for $n \neq 2q$, and

$$a_{2q} = \frac{1}{2\pi}\int_0^{\pi}\{\cos 4qt + 1\}dt = \frac{1}{2}.$$

Furthermore,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \cos 2qt\, \sin nt\, dt = \frac{1}{2\pi}\int_0^{\pi}\{\sin(n+2q)t + \sin(n-2q)t\}dt.$$

If $n \neq 2q$, then

$$\begin{aligned} b_n &= \frac{1}{2\pi}\left[-\frac{\cos(2q+n)t}{2q+n} - \frac{\cos(n-2q)t}{n-2q}\right]_0^{\pi} \\ &= \frac{1}{2\pi}\left\{-\frac{(-1)^{2q+n}}{2q+n} - \frac{(-1)^{n-2q}}{n-2q} + \frac{1}{2q+n} + \frac{1}{n-2q}\right\} \\ &= \frac{1}{2\pi}\left(\frac{1}{n+2q} + \frac{1}{n-2q}\right)\{1-(-1)^n\}. \end{aligned}$$

If $2n \neq 2q$, we immediately get $a_{2n} = 0$.
If $2n = 2q$, then

$$b_{2q} = \frac{1}{2\pi}\int_0^{\pi} \sin 4qt\, dt = \frac{1}{2\pi}\left[-\frac{1}{4q}\cos 4qt\right]_0^{\pi} = 0,$$

hence $b_{2n} = 0$ for every $n \in \mathbb{N}$.

Finally,

$$b_{2n-1} = \frac{1}{\pi}\left\{\frac{1}{(2n-1)+2q} + \frac{1}{(2n-1)-2q}\right\} = \frac{2}{\pi}\cdot\frac{2n-1}{(2n-1)^2-4q^2},$$

and the Fourier series becomes with an equality sign, cf. the beginning,

$$f^*(t) = \frac{1}{2}\cos 2qt + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2n-1}{(2n-1)^2-4q^2}\sin(2n-1)t.$$

2) Now, $2q \neq 0$, so applying **Parseval's equation**,

$$\frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(2n-1)^2}{\{(2n-1)^2-4q^2\}^2} = \frac{2}{2\pi}\int_0^{\pi}\cos^2 2qt\, dt = \frac{1}{2\pi}\cdot\pi = \frac{1}{2},$$

hence by a rearrangement

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2}{\{(2n-1)^2-4q^2\}^2} = \frac{\pi^2}{4}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi^2}{16}.$$

Alternatively, the latter sum can also be found in the following traditional way. It is, however, rather difficult, so I shall only sketch the solution in the following.

a) First by a decomposition,

$$\frac{(2n-1)^2}{\{(2n-1)^2-4q^2\}^2} = \left(\frac{2n-1}{(2n-1)^2-4q^2}\right)^2 = \left(\frac{1}{2}\left\{\frac{1}{2n-2q-1}+\frac{1}{2n+2q-1}\right\}\right)^2$$
$$= \frac{1}{4}\left\{\frac{1}{(2b-2q-1)^2}+\frac{1}{(2n+2q-1)^2}+\frac{1}{2q}\left(\frac{1}{2n-2q-1}-\frac{1}{2n+2q-1}\right)\right\}.$$

b) Then we obtain after a very long calculation that

$$s_n' = \sum_{n=1}^{N}\left(\frac{1}{2n-2q-1}-\frac{1}{2n+2q-1}\right) = -\sum_{p=N-q+1}^{N+q}\frac{1}{2p-1} \to 0 \qquad \text{for } N \to \infty,$$

because

$$\left|\sum_{p=N-q+1}^{N+q}\frac{1}{2p-1}\right| \le 2q \cdot \frac{1}{2N-2q+1} \to 0 \quad \text{for } N \to \infty.$$

(We have $2q$ terms which are all smaller than or equal to the first term).

c) Then it follows from (a) and (b) that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(2n-1)^2}{\{(2n-1)^2-4q^2\}^2}&=\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(2n-2q-1)^2}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(2n+2q+1)^2}\\&=\frac{1}{4}\sum_{p=-q+1}^{\infty}\frac{1}{(2p-1)^2}+\frac{1}{4}\sum_{p=q+1}^{\infty}\frac{1}{(2p-1)^2}=\cdots=\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^2}\\&=\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^2}\left\{1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\cdots\right\}\cdot\left[\sum_{p=0}^{\infty}\left(\frac{1}{4}\right)^p\right]^{-1}\\&=\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p^2}\cdot\left\{\frac{1}{1-\frac{1}{4}}\right\}^{-1}=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{\pi^2}{6}=\frac{\pi^2}{16}.\end{aligned}$$

Remark 3.1 For $q=0$ it follows immediately that

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2}{\{(2n-1)^2-4q^2\}^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=2\cdot\frac{\pi^2}{16}=\frac{\pi^2}{8},$$

and thus *not* $\dfrac{\pi^2}{16}$, which one might expect.

Example 3.5 *The periodic function $f:\mathbb{R}\to\mathbb{R}$ of period 2π is defined by*

$$f(t)=t^2,\qquad t\in\,]-\pi,\pi].$$

It can be proved that f has the Fourier series

(12) $$f\sim\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nt.$$

1) Prove that this Fourier series (12) is uniformly convergent, and find its sum function.

2) Prove by applying Parseval's equation that $\sum_{n=1}^{\infty}\dfrac{1}{n^4}=\dfrac{\pi^4}{90}$.

3) Prove that $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}=-\dfrac{\pi^2}{12}$.

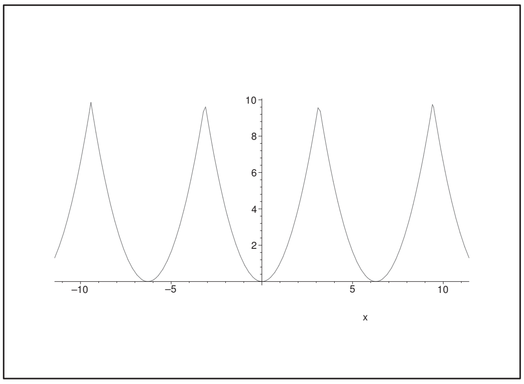
4) Find an integer N, such that $\left|-\dfrac{\pi^2}{12}-\sum_{n=1}^{N}\dfrac{(-1)^n}{n^2}\right|\le 10^{-4}$.

Introduction. The function is continuous and piecewise C^1 without vertical half tangents, hence $f\in K_{2\pi}^*$. The Fourier series is by the **main theorem** convergent with $f(t)$ itself as its sum function. Cf. the figure.

1) It follows from

$$\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}|(-1)^n\cos nt|\le\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}=\frac{\pi^2}{3}+4\cdot\frac{\pi^2}{6}=\pi^2<\infty,$$

that the Fourier series has a convergent majoring series, hence it is uniformly convergent with the sum function $f(t)$.



2) First we set up **Parseval's equation**:

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(t)]^2 dt.$$

It follows from the Fourier series that

$$\frac{1}{2}a_0 = \frac{\pi^2}{3}, \qquad \text{dvs. } a_0 = \frac{2}{3}\pi^2, \quad \text{og} \quad a_n = \frac{4(-1)^n}{n^2},\ n \in \mathbb{N}.$$

Then by an insertion,

$$\frac{1}{2}\left(\frac{2}{3}\pi\right)^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^4 dt = \frac{2}{5}\pi^4,$$

and thus by a rearrangement,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{16}\left\{\frac{2}{5}\pi^4 - \frac{4}{18}\pi^4\right\} = \frac{\pi^4}{8}\left(\frac{1}{5} - \frac{1}{9}\right) = \frac{\pi^4}{8}\cdot\frac{4}{45} = \frac{\pi^4}{90}.$$

3) If we put $t = 0$ into (12), we get since $f(t)$ is the sum function that

$$f(0) = 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

so by a rearrangement,

$$(13)\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

4) The series (13) is alternating. Since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ tends decreasingly towards 0, it follows from **Leibniz's criterion** that we have the following error estimate,

$$\left|-\frac{\pi^2}{12} - \sum_{n=1}^{N}\frac{(-1)^n}{n^2}\right| \leq \left|\frac{(-1)^{N+1}}{(N+1)^2}\right| = \frac{1}{(N+1)^2}.$$

Then we have

$$\frac{1}{(N+1)^2} \le 10^{-4} \le \frac{1}{100^2}$$

for $N+1 \ge 100$, so we get by the error estimate that we can choose $N \ge 99$.

Example 3.6 *Let $f \in K_{2\pi}$ be given by*

$$f(t) = \sin\frac{t}{2}, \qquad t \in\,]-\pi, \pi].$$

1) Sketch the graph of f in the interval $]-3\pi, 3\pi]$.

2) Find the Fourier series for f.
(Hint: It is given without proof that

$$\left.\int_0^{\pi} \sin\frac{t}{2}\sin nt\,dt = (-1)^{n-1}\cdot\frac{4n}{4n^2-1}, \qquad n \in \mathbb{N}\right).$$

3) Find the sum of the Fourier series at the point $t = \dfrac{7\pi}{3}$.

4) Explain why the Fourier series is not uniformly convergent.

5) Apply Parseval's equation in order to prove that

$$\sum_{n=1}^{\infty}\left(\frac{n}{4n^2-1}\right)^2 = \frac{\pi^2}{64}.$$

1) The function is odd and piecewise C^∞ without vertical half tangents, and with discontinuities at $t = (2p+1)\pi$, $p \in \mathbb{Z}$. It therefore follows from the **main theorem** that the Fourier series is convergent with the sum function

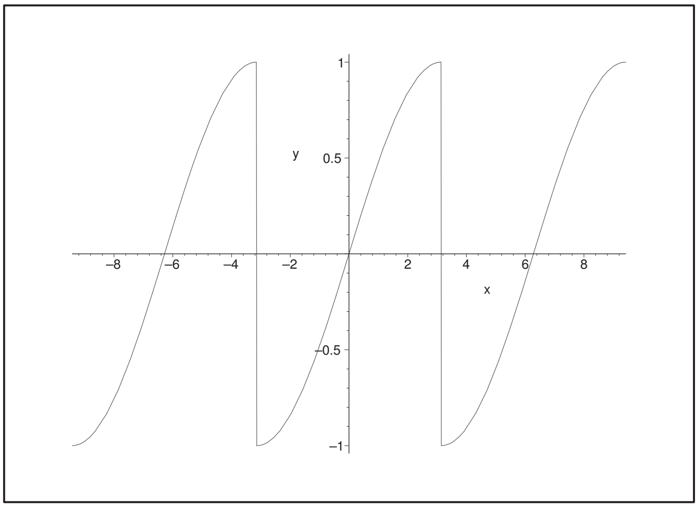
$$f^*(t) = \begin{cases} f(t) & \text{for } t \neq (2p+1)\pi, \quad p \in \mathbb{Z}, \\ 0 & \text{for } t = (2p+1)\pi, \quad p \in \mathbb{Z}. \end{cases}$$

2) The function f is odd, so $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin\left(\frac{t}{2}\right)\sin(nt)dt = \frac{2}{\pi}\cdot\frac{4n\cos(n\pi)}{1-4n^2}, \quad n \in \mathbb{N},$$

and the Fourier series is given with its sum function by

$$f^*(t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{1-4n^2}\sin(nt).$$



3) It follows from the periodicity that

$$f^*\left(\frac{7\pi}{3}\right) = f^*\left(\frac{7\pi}{3} - 2\pi\right) = f^*\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{6} = \frac{1}{2} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{1-4n^2}\,\sin\frac{n\pi}{3}.$$

4) The sum function $f^*(t)$ is *not* continuous, hence the convergence *cannot* be uniform. [In fact, if the convergence was uniform, then the sum function should be continuous, which it is not].

5) Then we get by **Parseval's equation**,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2\left(\frac{t}{2}\right)dt = 1 = \sum_{n=1}^{\infty} b_n^2 = \frac{64}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{n}{1-4n^2}\right)^2,$$

hence by a rearrangement,

$$\sum_{n=1}^{\infty}\left(\frac{n}{4n^2-1}\right)^2 = \frac{\pi^2}{64}.$$

4 Fourier series in the theory of beams

Example 4.1 *A periodic function f of period 2ℓ is given in the interval $]-\ell,\ell[$ by*

$$f(t)=\begin{cases} 0 & \text{for } -\ell<t<-\dfrac{\ell}{2},\\ -q_0 & \text{for } -\dfrac{\ell}{2}<t<0,\\ q_0 & \text{for } 0<t<\dfrac{\ell}{2},\\ 0 & \text{for } \dfrac{\ell}{2}<t\le\ell,\end{cases}$$

where q_0 is a positive constant.
We define in points of discontinuity $f(t)=\frac{1}{2}\{f(t+)+f(t-)\}$.

1) Sketch the graph of $f(t)$ in $]-\ell,\ell[$, and prove that the Fourier series for f is given by

$$\frac{2q_0}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}\sin(2n-1)\frac{\pi}{\ell}t+\frac{1-(-1)^n}{2n}\sin 2n\frac{\pi}{\ell}t\right].$$

2) A simply supported beam of length ℓ and of bending stiffness EI is loaded (constant load q_0 on the first half of the beam).

a) The linearized boundary value problem for the bending $u(x)$ of the beam by the load $q(x)$ is

$$\frac{d^4u}{dx^4}=\frac{q(x)}{EI},\qquad u(0)=u(\ell)=u''(0)=u''(\ell)=0.$$

b) Find the bending $u(x)$ in the form of a Fourier series of the type

$$\sum_{n=1}^{\infty}\left\{b_{2n-1}\sin(2n-1)\frac{\pi}{\ell}x+b_{2n}\sin 2n\frac{\pi}{\ell}x\right\},$$

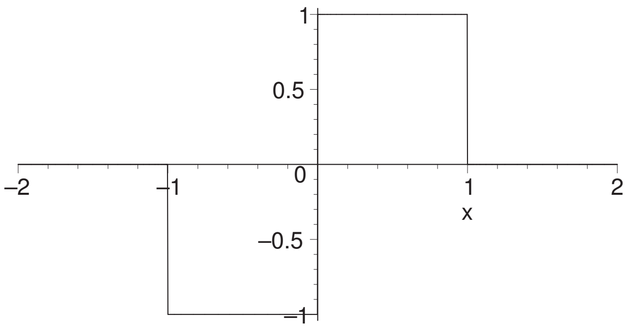
where the boundary value problem in (a) is solved by means of the method of Fourier series, and where the result of (1) is applied.

c) Prove that the bending $u\left(\frac{\ell}{2}\right)$ can be written as the series

$$\frac{2q_o\ell^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5},$$

and explain why the series is convergent. Find an approximative value of $u\left(\frac{\ell}{2}\right)$ with an error which is smaller than $\frac{2q_0\ell^4}{EI\pi^5}\cdot\frac{1}{7^5}$.

1) Clearly, $f\in K_{2\pi}^*$, because f is piecewise constant. The function f is already adjusted. and since f is *odd*, we get $a_n=0$, and the Fourier series is a sine series, which by the **main theorem** has the sum function $f(t)$.



By a calculation,

$$\begin{aligned} b_n &= \frac{2}{\ell}\int_0^{\ell} f(t)\sin\frac{n\pi t}{\ell}\,dt = \frac{2q_0}{\ell}\int_0^{\ell/2}\sin\left(\frac{n\pi}{\ell}\right)dt \\ &= \frac{2q_0}{\ell}\cdot\frac{\ell}{n\pi}\left[-\cos\left(\frac{n\pi t}{\ell}\right)\right]_0^{\ell/2} = \frac{2q_0}{n\pi}\left\{1-\cos\left(\frac{n\pi}{2}\right)\right\}, \end{aligned}$$

so

$$b_{2n-1} = \frac{2q_0}{\pi}\cdot\frac{1}{2n-1} \quad \text{og} \quad b_{2n} = \frac{2q_0}{\pi}\cdot\frac{1-(-1)^n}{2n}.$$

The Fourier series is with an equality sign, cf. the beginning, given as required by

$$f(t)=\frac{2q_0}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{2n-1}\sin(2n-1)\frac{\pi}{\ell}t+\frac{1-(-1)^n}{2n}\sin 2n\frac{\pi}{\ell}t\right\}.$$

2) Now, $q(x)=f(x)$, so it follows from (1) that

$$\frac{q(x)}{EI}=\frac{2q_0}{\pi EI}\left\{\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\frac{\pi}{\ell}t+\frac{1-(-1)^n}{2n}\sin 2n\frac{\pi}{\ell}t\right\}.$$

This series is *not uniformly convergent* (because $q(x)$ is not continuous), so strictly speaking we are not allowed to perform termwise integration. Nevertheless, *if* we perform termwise integration four times, we get

$$u(t)=\frac{2q_0\ell^4}{\pi^5 EI}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin(2n-1)\frac{\pi}{\ell}t+\frac{1-(-1)^n}{(2n)^5}\sin 2n\frac{\pi}{\ell}t\right\}+c_0+c_1t+c_2t^2+c_3t^3,$$

where we later shall come back to this inconsistency with the usual theory.

Then by the boundary conditions,

$$u(0)=0=c_0 \quad\text{and}\quad u(\ell)=0=\ell(c_1+c_2\ell+c_3\ell^2),$$

$$u''(0)=0=2c_2 \quad\text{and}\quad u''(\ell)=0=2c_2+6c_3\ell,$$

hence $c_0=c_1=c_2=c_3=0$. Thus

$$u(t)=\frac{2q_0\ell^4}{\pi^5 EI}\left\{\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin(2n-1)\frac{\pi}{\ell}t+\frac{1-(-1)^n}{(2n)^5}\sin 2n\frac{\pi}{\ell}t\right\}.$$

If $t=\dfrac{\ell}{2}$, then $\sin\left(2n\dfrac{\pi}{\ell}\cdot\dfrac{\ell}{2}\right)=0$ and $\sin\left((2n-1)\dfrac{\pi}{\ell}\cdot\dfrac{\ell}{2}\right)=(-1)^{n+1}$, hence

$$u\left(\frac{\ell}{2}\right)=\frac{2q_0\ell^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5}.$$

This series obviously has the convergent majoring series

$$\frac{2q_0\ell^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{1}{n^5}.$$

The series is *alternating* and $\dfrac{1}{(2n-1)^5}$ is decreasing, so we get the estimate

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5}-\sum_{n=1}^{3}\frac{(-1)^{n+1}}{(2n-1)^5}\right|\le\frac{1}{(2\cdot 4-1)^5}=\frac{1}{7^5},$$

whence,

$$u\left(\frac{\ell}{2}\right)\approx\frac{2q_0\ell^4}{EI\pi^5}\left\{\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}\right\}=\frac{1512986}{759375\pi^5}\cdot\frac{q_0\ell^4}{EI}\approx 0,006511\cdot\frac{q_0\ell^4}{EI}.$$

We can now repair the "hole" in the argument above by directly solve the equation

$$\frac{d^4u}{dx^4} = \frac{q(x)}{EI}, \qquad u(0) = u(\ell) = u''(0) = u''(\ell) = 0$$

with four (difficult) integrations. First we get from

$$\frac{d^4u}{dx^4} = \begin{cases} \dfrac{q_0}{EI} & \text{for } x \in \left]0, \dfrac{\ell}{2}\right[, \\[2ex] 0 & \text{for } x \in \left]\dfrac{\ell}{2}, \ell\right[, \end{cases}$$

that

$$\frac{d^3u}{dx^3} = \begin{cases} \dfrac{q_0}{EI}\, x + c_1 & \text{for } x \in \left[0, \dfrac{\ell}{2}\right], \\[2ex] \dfrac{q_0\ell}{2EI} + c_1 & \text{for } x \in \left[\dfrac{\ell}{2}, \ell\right], \end{cases}$$

hence

$$\frac{d^2u}{dx^2} = \begin{cases} \dfrac{q_0}{2EI}\, x^2 + c_1 + c_2, & \text{for } x \in \left[0, \dfrac{\ell}{2}\right], \\[2ex] \dfrac{q_0\ell^2}{8EI} + \dfrac{c_1\ell}{2} + c_2 + \left(\dfrac{q_0\ell}{2EI} + c_1\right)\left(x - \dfrac{\ell}{2}\right), & \text{for } x \in \left[\dfrac{\ell}{2}, \ell\right]. \end{cases}$$

Now, $u''(0) = 0$ so $c_2 = 0$, and since $u''(\ell) = 0$ we get

$$0 = \frac{q_0\ell^2}{8EI} + \frac{c_1\ell}{2} + \frac{q_0\ell^2}{4EI} + \frac{c_1\ell}{2} = \frac{3q\ell^2}{8EI} + c_1\ell,$$

thus

$$c_1 = -\frac{3q_0\ell}{8EI}.$$

By insertion and reduction,

$$\frac{d^2u}{dx^2} = \begin{cases} \dfrac{q_0}{2EI}\, x^2 - \dfrac{3q_0\ell}{8EI}\, x & \text{for } 0 \le x \le \dfrac{\ell}{2}, \\[2ex] -\dfrac{q_0\ell^2}{16EI} + \dfrac{q_0\ell}{8EI}\left(x - \dfrac{\ell}{2}\right) & \text{for } \dfrac{\ell}{2} < x \le \ell, \end{cases}$$

so

$$\frac{du}{dx} = \begin{cases} \dfrac{q_0}{6EI}\, x^3 - \dfrac{3q_0\ell}{16EI}\, x^2 + c_3, & \text{for } 0 \le x \le \dfrac{\ell}{2}, \\[2ex] \dfrac{q_0\ell^3}{48EI} - \dfrac{3q_0\ell^3}{64EI} + c_3 - \dfrac{q_0\ell^2}{16EI}\left(x - \dfrac{\ell}{2}\right) + \dfrac{q_0\ell}{16EI}\left(x - \dfrac{\ell}{2}\right)^2, & \text{for } \dfrac{\ell}{2} < x \le \ell. \end{cases}$$

Since $u(0)=0$, we therefore get for $0 \le x \le \dfrac{\ell}{2}$ that

$$u(x) = \frac{q_0 x^4}{24EI} - \frac{q_0 \ell x^3}{16EI} + c_3 x \qquad \text{for } 0 \le x \le \frac{\ell}{2}.$$

If then $x \in \left[\dfrac{\ell}{2}, \ell\right]$ we have

$$\begin{aligned} u(x) &= \frac{q_0\ell^4}{24\cdot 16EI} - \frac{q_0\ell^4}{16\cdot 8EI} + c_3\frac{\ell}{2} + \left(c_3 - \frac{5q_0\ell^3}{192EI}\right)\left(x-\frac{\ell}{2}\right) \\ &\quad - \frac{q_0\ell^2}{32EI}\left(x-\frac{\ell}{2}\right)^2 + \frac{q_0\ell}{48EI}\left(x-\frac{\ell}{2}\right)^3, \end{aligned}$$

where

$$\begin{aligned} u(\ell) = 0 &= \frac{q_0\ell^4}{16EI}\left\{\frac{1}{24} - \frac{1}{8} - \frac{1}{2}\cdot\frac{5}{12} - \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{8\cdot 3}\right\} + c_3\ell \\ &= \frac{q_0\ell^4}{16EI}\cdot\frac{1-3-5-3+1}{24} + c_3\ell = -\frac{3q_0\ell^4}{128EI} + c_3\ell, \end{aligned}$$

so

$$c_3 = \frac{3}{128}\cdot\frac{q_0\ell^4}{EI}.$$

Then by insertion of c_3 and some further calculations we finally obtain that

$$u(x) = \begin{cases} \dfrac{q_0}{24EI}\,x^4 - \dfrac{q_0\ell}{16EI}\,x^3 + \dfrac{3}{128}\,\dfrac{q_0\ell^3}{EI}\,x, & \text{for } 0 \le x \le \dfrac{\ell}{2}, \\[2ex] \dfrac{7}{384}\cdot\dfrac{q_0\ell^3}{EI}(\ell - x) - \dfrac{1}{48}\cdot\dfrac{q_0\ell}{EI}(\ell-x)^3, & \text{for } \dfrac{\ell}{2} \le x \le \ell, \end{cases}$$

where it is more convenient for $x \in \left[\dfrac{\ell}{2}, \ell\right]$ to use $\ell - x$ as the variable.

If $x = \dfrac{\ell}{2}$, then

$$u\left(\frac{\ell}{2}\right) = \frac{5}{768}\cdot\frac{q_0\ell^4}{EI} = \frac{2q_0\ell^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^5},$$

hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5}{768}\cdot\frac{\pi^5}{2} = \frac{5\pi^5}{1536}.$$